has become the first part of a two volume treatise. The second volume is intended to cover the work on the quantised systems up to the current time.

ROGER DODD
HIROSHIMA UNIVERSITY

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Moduli of smoothness, by Z. Ditzian and V. Totik. Springer Series in Computational Mathematics, Springer-Verlag, New York, Berlin and Heidelberg, 1987, ix+225 pp., $54.90. ISBN 0-387-96536-x

Moduli of smoothness play a basic role in approximation theory, Fourier analysis and their applications. For a given function f, the domain of which is a (bounded) interval D, they essentially measure the structure or smoothness of f via the rth (symmetric) difference

$$\Delta_h^r f(x) := \sum_{k=0}^{r} \binom{r}{k} (-1)^k f(x + rh/2 - kh)$$

(with the convention that $\Delta_h^r f(x) = 0$ if $x \pm rh/2 \notin D$). In fact, for functions f belonging to the Lebesgue space $L^p(D)$, $1 \leq p < \infty$, or the space $C(D)$ $(p = \infty)$ of continuous functions, the classical rth modulus

$$\omega^r(f,t)_p := \sup_{|h| \leq t} \|\Delta_h^r f\|_p \tag{1}$$

has turned out to be a rather good measure for determining the rate of convergence of best approximation or of particular linear approximation processes.

For example, for 2π-periodic functions f, D. Jackson (1911) and S. N. Bernstein (1912) showed that the error of best approximation $E_n^*(f)_p$ by trigonometric polynomials of degree at most n has the same rate of convergence as the rth modulus in the sense that, for $0 < \alpha < r$,

$$\omega^r(f,t)_p = \mathscr{O}(t^\alpha) \qquad (t \to 0) \Leftrightarrow E_n^*(f)_p = \mathscr{O}(n^{-\alpha}) \qquad (n \to \infty). \tag{2}$$

In the case of algebraic approximation, however, that is by algebraic polynomials $p_n \in \mathscr{P}_n$ of degree at most n, this result is no longer true. Though one has here the direct estimate

$$E_n(f)_p := \inf_{p_n \in \mathscr{P}_n} \|f - p_n\|_p \leq K\omega^r\left(f, \frac{1}{n}\right)_p,$$

given in the doctoral thesis of D. Jackson (1911), it was observed by S. M. Nikolskii (1946) that for functions f satisfying

$$\omega^r(f,t)_p = \mathscr{O}(t^\alpha) \qquad (0 < \alpha < r), \tag{3}$$

the polynomial $p_n^* \in \mathscr{P}_n$ of best approximation has a faster rate of convergence near the boundary of D than in the interior. In fact, it was the Russian school in approximation, in particular A. F. Timan (1951) and V. K. Dzjadyk (1959), that succeeded in characterizing (3) in terms of algebraic polynomials for

$p = \infty$ and $D = [-1,1]$. They showed that (3) is equivalent to the existence of a sequence of $p_n \in \mathscr{P}_n$ such that

$$(4) \qquad |f(x) - p_n(x)| = \mathscr{O}(\delta_n^\alpha(x)), \delta_n(x) := \left(\varphi(x) + \frac{1}{n}\right) \Big/ n$$

uniformly in $x \in [-1,1]$, where they brought in the weight $\varphi(x) := \sqrt{1-x^2}$.

However, for only integrable functions $f \in L^p(-1,1)$, $1 \le p < \infty$, (4) has no longer sense; even

$$\inf_{p_n \in \mathscr{P}_n} \|\delta_n^{-\alpha}[f - p_n]\|_p = \mathscr{O}(1)$$

is not equivalent to (1) as pointed out by V. P. Motornii (1971) and R. DeVore (1978). The reason may be due to the fact that one should not consider (4) but

$$(5) \qquad E_n(f)_p = \mathscr{O}(n^{-\alpha}) \qquad (0 < \alpha < r)$$

and characterize it by a suitable modification of the classical modulus. This long-outstanding problem was only solved in 1980 by two schools of approximation. Thus K. G. Ivanov (announced in 1980, proofs followed only in 1983) modified the so-called τ-modulus, developed and studied intensively by the Bulgarian school (e.g., B. L. Sendov (1968), A. Andreev, V. A. Popov), to the weighted one

$$\tau_\varphi^r\left(f, \frac{1}{n}\right)_p := \left\|\omega_\varphi^r\left(f, \cdot, \frac{1}{n}\right)\right\|_p$$

with local modulus ($x \in [-1,1]$)

$$\omega_\varphi^r\left(f, x, \frac{1}{n}\right) := \frac{1}{2\delta_n(x)} \int_{-\delta_n(x)}^{\delta_n(x)} |\Delta_u^r f(x)|\, du.$$

Thus, (5) is equivalent to

$$(6) \qquad \tau_\varphi^r(f, 1/n)_p = \mathscr{O}(n^{-\alpha}) \qquad (0 < \alpha < r).$$

In the alternative approach by Butzer, Stens, Wehrens (1976/1980) the classical translation $f(x+h)$ is replaced by the so-called Legendre translation

$$(\tau_h^L f)(x) := \frac{1}{\pi} \int_{-1}^{1} f(xh + u\sqrt{(1-x^2)(1-h^2)}) \frac{du}{\sqrt{1-u^2}} \qquad (x, h \in [-1,1]);$$

it has the same properties with respect to the Legendre transform (e.g., multiplier operator) as has $f(x+h)$ with respect to the finite Fourier transform. In this frame they defined the rth modulus of continuity by

$$\omega_L^r(f,t)_p := \sup_{1-t \le h_j \le 1} \left\|\Delta_{h_1}^L \Delta_{h_2}^L \cdots \Delta_{h_r}^L f\right\|_p$$

with difference $\Delta_h^L f := \tau_h^L f - f$, and established not only the complete analogue of the Jackson-Bernstein result (2) for $E_n(f)_p$ but also Steckin and Zamansky-type estimates. In (2) one just has to replace ω_r by ω_r^L, α by 2α and $t \to 0+$ by $t \to 1-$. (See also Stens [**2**].) The case of weighted L^p-spaces with Jacobi weights can be handled by using a Jacobi translation instead.

Apart from the theory of best approximation, another field in the broad area of approximation theory is the study of various linear approximation processes together with characterizations of their rate of convergence. For example, for the classical Bernstein polynomials $B_n f$, defined for $f \in C[0,1]$, H. Berens and G. G. Lorentz (1972) (and independently R. DeVore) established the equivalence theorem ($0 < \alpha < 1$, cf. (3,4)),

$$\omega^2(f,t)_\infty = \mathscr{O}(t^{2\alpha}) \Leftrightarrow |B_n f(x) - f(x)| = \mathscr{O}((\varphi^2(x)/n)^\alpha), \tag{7}$$

uniformly for $x \in [0,1]$ with weight $\varphi(x) = \sqrt{x(1-x)}$. The essential point of the proof was the characterization of the behaviour of $B_n f$ in terms of suitable Peetre-K-functionals, e.g.,

$$K_\varphi^2(f,t) := \inf\{\|f-g\|_\infty + t\|\varphi^2 g''\|_\infty\}, \tag{8}$$

where the infimum is taken over all $g \in C[0,1]$ which are twice differentiable on $(0,1)$ such that $\varphi^2 g'' \in C[0,1]$. Such K-functionals were investigated by Z. Ditzian (1980), also for L^p-spaces; he introduced the associated weighted modulus of smoothness with main part

$$\omega_\varphi^r(f,t)_p := \sup_{|h|\le t} \|\Delta_{h\varphi(x)}^r f(x)\|_p \tag{9}$$

(the interval in question being related to the weight φ) which turned out to be equivalent to the K-functional (see especially H. Johnen-K. Scherer (1976) for the case $\varphi(x) = 1$, i.e., (1)). Indeed, he proved for $p = \infty$ a result similar to (5,6), namely ($\varphi(x) = \sqrt{x(1-x)}$),

$$\omega_\varphi^2(f,t)_\infty = \mathscr{O}(t^{2\alpha}) \Leftrightarrow \|B_n f - f\|_\infty = \mathscr{O}(n^{-\alpha}). \tag{10}$$

Observe that the weight φ, coupled in (7) with the rate of $B_n f$, is here coupled with the modulus instead.

At this point, solutions for basic analogous problems in the space L^p, thus, for example, for the Kantorovich version $K_n f$ of $B_n f$, were still open. After initial work by W. Hoeffding (1971), V. Maier (1978), S. D. Riemenschneider (1978), M. Becker and R. J. Nessel (1978/1980), it was V. Totik who, in a series of 14 papers since 1983, succeeded in characterizing the error in L^p, $1 \le p < \infty$, for $K_n f$ as well as for the counterparts of Szász-Mirakjan, Baskakov, Meyer-König and Zeller and Gamma operators by making use of the weighted modulus (9). As the reviewers just learned, Zhou Xin-long [**3**, **4**] solved some of these problems for the Kantorovich version of the Bernstein, Szász-Mirakjan and Baskakov operators simultaneously and independently.

Let us now turn to the book by Ditzian and Totik under review. It is divided into two parts, the first dealing with weighted moduli of smoothness of type (9) on (not necessarily bounded) intervals on **R** with or without weighted norms, while the second one concerns applications. The main material of part one is devoted to the equivalence between ω_φ^r and K-functionals like (8) with arbitrary φ. This is one of the key and difficult results of the book—it requires some involved technical proofs.

This part is rounded off by basic properties of the weighted modulus similar to the classical one, including the Marchaud inequality.

Part two starts off with the characterization of best (weighted) algebraic approximation on the interval $[-1,1]$ in terms of ω_φ^r. It is closely related to the trigonometric case; results analogous to the Jackson inequality, valid for 2π-periodic functions,

$$E_n^*(f)_p \leq C\omega^r \left(f, \frac{1}{n}\right)_p \tag{11}$$

and the weak-type inequality

$$\omega^r \left(f, \frac{1}{n}\right)_p \leq Cn^{-r} \sum_{k=0}^{n} (k+1)^{r-1} E_k^*(f)_p, \tag{12}$$

due to S. B. Steckin (1951), are established. Also other expedient tools are developed here, such as a Bernstein-type inequality in weighted norms or estimates for the derivatives of the polynomial of best approximation. Concerning applications to positive operators, the authors considered the class of Bernstein-type or exponential-type operators and their Kantorovich versions, including linear combinations of these. Here the usefulness of ω_φ^r becomes very evident since it fits elegantly in the derivation of the direct (cf. (11)) as well as associated inverse theorems (cf. (12)), extending the weak-type estimates of E. van Wickeren (1986/1987). Finally, weighted polynomial approximation in $L^p(\mathbf{R})$ is discussed (using the G. Freud approach and very recent results of H. N. Mhaskar (1986) and A. L. Levin-D. S. Lubinsky (1987)), as well as polynomial approximation in several variables on cubes and polytopes. The book is rounded up with a juxtaposition of ω_φ^r to moduli of smoothness studied by Ivanov, by Butzer-Stens-Wehrens and by M. K. Potapov (1981/1983). However, the different moduli are not compared explicitly (which may be very difficult). In this respect Ivanov [**1**] remarks (without proof) that his modulus τ_φ^r is equivalent to the Ditzian-Totik modulus ω_φ^r. In any case, their modulus seems to be the most natural one as has been demonstrated in a masterful way in their book.

The book itself has to be seen as a continuation of foregoing individual work of the two authors. Though essentially self-contained, the reader should possess good knowledge of trigonometric and algebraic approximation in order to be able to understand the problems which are here solved via ω_φ^r. The reason may be found in the remark at the end of the introduction: "This book was written as a research paper, but it has outgrown this description in size. However, it has retained the characteristics of a research work and all of the results presented are new."

Nevertheless, the main and difficult problems of algebraic approximation are presented and solved here, including a number of tools which should be useful for later more practical applications. The proofs do not involve "soft" analysis but most of them consist of really "hard" analysis; some of them tend to be somewhat technical. Unfortunately, the motivations and remarks are often rather short. The index is only half a page long, and so not very helpful. However, the list of symbols is handy and the list of references is excellent and includes all relevant titles in the field. In regard to the title of the book, it is a little misleading since only a special but new kind of modulus is discussed; no other types of moduli, for example those in more abstract spaces, are considered. A more appropriate title would have been *Moduli of smoothness with weights in algebraic approximation.*

All in all, this book is a valuable addition to the literature on approximation written by Edmonton's highly experienced analyst Zeev Ditzian and Vilmost Totik, one of Hungary's strongest young analysts.

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PAUL L. BUTZER
ERICH VAN WICKEREN
AACHEN UNIVERSITY OF TECHNOLOGY