Daverman's book is the first devoted exclusively to the theory of decompositions. It is much needed, and provides an excellent treatment of a subject of growing importance.

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Hamiltonian methods in the theory of solitons, by L. D. Faddeev and L. A. Takhtajan. Translated by A. G. Reyman. Springer-Verlag, Berlin, Heidelberg and New York, 1987, ix+592 pp., $110.00. ISBN 3-540-15579-1

The modern theory of integrable or solvable systems was initiated by the discoveries of Gardner, Greene, Kruskal, Miura and Zabusky in their investigations of the Korteweg-de Vries equation during the sixties. There then followed a period of intensive activity, which lasted until the late seventies, during which the characteristic features of these systems were explored and a vast class of such equations discovered. It is fair to say that many of the major advances in this field are associated with groups of researchers at a particular institute, such as the Leningrad group to which the authors of this book belong.

Most of the solvable equations possess a family of special solutions, which can be obtained in closed form. In the simplest cases, such as the Korteweg-de Vries equation, they can be given a physical interpretation as a collection of interacting particles. Each particle has only nearest neighbour interaction and the particles emerge from collision with one another undergoing only internal changes of phase, but without loss of energy. These special solutions or solitons (a term coined by the Princeton group) are connected with a group theoretic interpretation of the equations, which is due to the Kyoto school of researchers (Date, Jimbo, Kashiwara, Miwa, Sato and Sato). Essentially the soliton and rational solutions of a given equation are found to be obtained by solving a Hirota equation. This equation defines an algebraic variety which is generated by the orbit of a highest weight vector in an irreducible highest weight (level 1) representation of an affine Lie algebra under the action of the corresponding group. Any solution to the equation is called a τ-function; it can be considered as a generalisation of a θ-function. This work was begun in the eighties and is still proceeding.

One of the unsatisfactory features of the present theory is that it really only covers physical systems in two dimensions (one of which may be time). An exception (there are others) is the Kadomtsev-Petviashvili equation (2 space, 1 time). This has special solutions which have the form of a wave of infinite length with a cross-section given by the soliton solution of the Korteweg-de Vries equation. It is remarkable that the group theoretic treatment due to the Kyoto school of the Kadomtsev-Petviashvili equation has recently been given another interpretation in string theory. Sato and co-workers at Kyoto have recently attempted to extend their theory to higher dimensions by generalising their original work. This has resulted in the theory of D-modules.

The two authors of the volume under review belong to the Leningrad group of researchers (Faddeev, Korepin, Kulish, Reyman, Reshetikhin, Semenov-Tian-Shansky, Sklyanin and Smirnov). They have principally worked on the interpretation of solvable equations as classical fields and their quantisation. Naturally the interpretation of the classical field equations as Hamiltonian systems is the first step in such a program and the group have made extensive original contributions in this area. If the solutions of the solvable equations are required to satisfy a rapid decrease (Schwartz) condition at spatial infinity then the field equations can be interpreted as completely integrable Hamiltonian systems; there exists a canonical transformation to action-angle variables in which the equations of motion can be explicitly integrated. Solvable equations arise in hierarchies of increasing nonlinearity and order and in the Hamiltonian picture the members of a hierarchy are in involution with respect to one another. The definition of the Poisson structure for the solvable equation is usually straightforward. The Korteweg-de Vries equation is an exception here, and this is one of the reasons why the authors have settled for the nonlinear Schrödinger equation, another "universal" solvable equation, as their main equation for study.

A major innovation of the Leningrad group has been the derivation of the Poisson brackets satisfied by the transformed canonical variables through the study of the transition operator. To define this it is necessary to use the zero curvature representation of the solvable equation. This is essentially a system of linear equations involving a spectral parameter whose complete integrability, subject to the invariance of the spectral parameter, is guaranteed by the solvable equation being satisfied. The Poisson brackets for the transition operator at different values of the spectral parameter are given in terms of the two operators and a new entity, the classical r-matrix, which depends upon the two spectral values. By reason of the Jacobi identity the r-matrix satisfies an equation called the classical Yang-Baxter equation. Conversely if an r-matrix satisfies the classical Yang-Baxter equation it is possible to define a corresponding Poisson structure for the original field equation.

In the quantised version of the field equations the operator corresponding to r satisfies the Yang-Baxter equation (the classical Yang-Baxter equation is obtained from the $O(h)$ expansion of this equation). It is a remarkable fact that this equation also occurs in statistical mechanics. The analysis of the classical Yang-Baxter equation has been investigated mainly by the Russians (Belavin and Drinfeld in particular) whereas the Yang-Baxter equations have been studied principally by the Leningrad and Kyoto groups. There is a deep relationship with the theory of affine Lie algebras and the work on theta functions and modular forms due to Kac and Peterson.

The book is a treatise on the application of Hamiltonian methods to solvable equations in one space one time, treated as classical field equations. It is divided into two parts. The first part deals principally with the nonlinear Schrödinger equation as a basic example and the second part applies the methods to other interesting systems which include the Heisenberg ferromagnet, the sine-Gordon equation and also some lattice models such as the Toda lattice. The Hamiltonian structure of the equations is studied under a variety of boundary conditions; in addition to the Schwartz condition mentioned earlier the quasi-periodic and finite density boundary conditions are also considered. A nice feature of the book is a thorough working out of the relationship between the Hamiltonian techniques and other methods of analysis used to study solvable equations such as the Riemann-Hilbert method. A classification of integrable models based on the concept of the r-matrix is also presented at the end of the second part of the book.

The book does not include any work on systems in more than one space dimension such as the Kadomtsev-Petviashvili equation mentioned earlier. The complete integrability or solvability, that is the derivation of the equations of motion in the action angle variables and their explicit solution, is investigated in detail in the rapid decrease and finite density cases. The book does not cover the theory for the quasi-periodic situation.

There are detailed notes at the end of each chapter which give interesting background material, references to areas in which the equations arise, and brief summaries on, and references to, aspects of the theory which are not covered in the text. There is an extensive bibliography at the end of each chapter which is especially good on the Russian literature. It is well written and the ideas are clearly expressed. The book uses classical analysis; the techniques of global analysis which are needed for full mathematical rigor are deliberately avoided, as the authors believe this approach obscures the main developments in the theory and their interrelation with other branches of mathematics such as the theory of Lie groups.

The book covers the Hamiltonian theory of classical integrable equations up to the beginning of the eighties. As the authors state in their introduction, the original idea was to write a text on the quantum R-method, and this volume has become the first part of a two volume treatise. The second volume is intended to cover the work on the quantised systems up to the current time.

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Moduli of smoothness, by Z. Ditzian and V. Totik. Springer Series in Computational Mathematics, Springer-Verlag, New York, Berlin and Heidelberg, 1987, ix+225 pp., $54.90. ISBN 0-387-96536-x

Moduli of smoothness play a basic role in approximation theory, Fourier analysis and their applications. For a given function f, the domain of which is a (bounded) interval D, they essentially measure the structure or smoothness of f via the rth (symmetric) difference

$$\Delta_h^r f(x) := \sum_{k=0}^{r} \binom{r}{k} (-1)^k f(x + rh/2 - kh)$$

(with the convention that $\Delta_h^r f(x) = 0$ if $x \pm rh/2 \notin D$). In fact, for functions f belonging to the Lebesgue space $L^p(D)$, $1 \leq p < \infty$, or the space $C(D)$ $(p = \infty)$ of continuous functions, the classical rth modulus

$$(1) \qquad \omega^r(f,t)_p := \sup_{|h|\leq t} \|\Delta_h^r f\|_p$$

has turned out to be a rather good measure for determining the rate of convergence of best approximation or of particular linear approximation processes.

For example, for 2π-periodic functions f, D. Jackson (1911) and S. N. Bernstein (1912) showed that the error of best approximation $E_n^*(f)_p$ by trigonometric polynomials of degree at most n has the same rate of convergence as the rth modulus in the sense that, for $0 < \alpha < r$,

$$(2) \qquad \omega^r(f,t)_p = \mathscr{O}(t^\alpha) \qquad (t \to 0) \Leftrightarrow E_n^*(f)_p = \mathscr{O}(n^{-\alpha}) \qquad (n \to \infty).$$

In the case of algebraic approximation, however, that is by algebraic polynomials $p_n \in \mathscr{P}_n$ of degree at most n, this result is no longer true. Though one has here the direct estimate

$$E_n(f)_p := \inf_{p_n \in \mathscr{P}_n} \|f - p_n\|_p \leq K\omega^r \left(f, \frac{1}{n}\right)_p,$$

given in the doctoral thesis of D. Jackson (1911), it was observed by S. M. Nikolskii (1946) that for functions f satisfying

$$(3) \qquad \omega^r(f,t)_p = \mathscr{O}(t^\alpha) \qquad (0 < \alpha < r),$$

the polynomial $p_n^* \in \mathscr{P}_n$ of best approximation has a faster rate of convergence near the boundary of D than in the interior. In fact, it was the Russian school in approximation, in particular A. F. Timan (1951) and V. K. Dzjadyk (1959), that succeeded in characterizing (3) in terms of algebraic polynomials for