[W] Hermann Weyl, *Inequalities between the two kinds of eigenvalues of a linear transformation*, Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 408–411.

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In the summer of 1959 I visited the Mathematics Department in Berkeley for the first time. There was a summer-long seminar in functional analysis that year, and practically everyone working in Banach algebras and associated topics spent some time there. I met Errett Bishop at one of the post-seminar teas, and we talked of problems in several complex variables related to function algebras. He asked me whether or not I knew if any analytic polyhedron in a complex manifold of dimension n could be approximated by one defined by only n functions. I hadn't a clue, and I asked him why he supposed such a thing should be true. His answer: "Well, for a projective variety of dimension n, it is true that almost every projection to P^n is of degree n, and that's really the same thing for this special case." I was unable to see the connection, and, being a brash upstart fresh out of MIT, I assumed that there wasn't one. We passed on to a discussion of minimal boundaries; that spring I had studied his paper on this subject and was very impressed by the appearance of "hard" analysis in what I thought was a subject in "soft" analysis. I wanted to calculate the minimal boundary of analytic polyhedra. Errett instantly knew where I was stuck, and suggested that I needed to understand better how to cut down representing measures using peak sets.

Several months later, while writing up a set of notes on analytic spaces, I finally understood Bishop's connection between generic projections of projective varieties and (what later became known as) special analytic polyhedra. I discovered how easy it was to extend known theorems for the polydisc to analytic covers of the polydisc, and that his idea, if true, amounted to the assertion that any Stein space could be approximated by analytic covers of polydiscs! What a potent tool! All theorems proven locally for analytic spaces by means of the parametrization theorem could now be proven for arbitrarily large domains on Stein analytic spaces. I called him to tell him about my discovery. He seemed pleased and interested and added this: "Furthermore, if I can see how to convert an almost proper map to a proper one, all these theorems will extend to the whole Stein space, and furthermore, provide an embedding into C^n." "You mean you can prove Remmert's theorem?" I asked. He replied that he thought so.

In that way began one of the most important mathematical friendships of my career. For most of the following two years, Errett Bishop was a member of the Institute for Advanced Study, and I was an assistant professor at Princeton

University. We spent almost every day together discussing complex analysis, function algebras, and algebraic geometry. At times Errett would say things that assumed connections I had no idea how to establish. Some time later I would grasp the significance; once understood, the clarity of his vision would almost frighten me. Those visions provided me with deep insights into my own work, and often directed me to the solutions of my problems. Other times Errett would present a detailed proof to me: the beauty of his logical precision, and the simplicity of his exposition were striking. In this time I also learned that when Errett said that he thought he could prove something, he meant that he could prove it, but that he was not yet happy with the exposition.

Those days are long gone, and now I am writing a review of his *Selected Papers*. These papers have been imprinted in my memory for over twenty years and it is difficult now to reread them in an attempt to recreate the freshness and vitality they once had for me. There is an excellent description of the mathematics of Errett Bishop in the essay in this volume by John Wermer. The collection of essays in *Errett Bishop: Reflections on Him and His Research* [**EB**], also give a valuable account of his work. I cannot improve on those articles. Instead, I shall concentrate on the impact of Errett Bishop's ideas and way of thinking.

In the sequel I shall be referring to articles which are in this volume as well as elsewhere. Numbered references are to the bibliography in *Selected Papers* (page xxiii), and letter references are to the bibliography to follow. Originally I compiled a bibliography of works by other authors that were significantly influenced by Bishop's work. When that reached eight pages, I decided instead for a list of works to which reference was made in the text; the result therefore appears to be a random selection from what was first intended. I shall quote facts and theorems in an attempt to give a flavor of what is being asserted, and without proper regard to mathematical accuracy; I hope that such indiscretions will not offend the experts.

1. Approximation theorems on Riemann surfaces. Let S be a Riemann surface and R an algebra of holomorphic functions on S. Let us assume that S is R-holomorphically convex; i.e. (although this is not the description used by Bishop), for any discrete set D on S, there is an f in R such that $f(D)$ is a discrete set in the plane. It is an easy consequence of Cartan's Theorem B that there is an analytic space S' and a map of S onto S' which makes just the point identifications made by R so that R appears as the algebra of holomorphic functions on S'. This is true in all dimensions, and a proof of the more general result appears in [**16**] and [**R**]. We shall return to this below. My point here is that in 1957 Theorem B was not the household word it had become by the early '60s, and in [**2**] Bishop gives a function-theoretic proof of this result. His idea (one already put to good use in another sequence of papers by John Wermer [**W1**, **W2**]) was to concentrate on the measures on compact subsets of S annihilating R. The characterization of such measures became a central theme of Errett's work; in fact, in the whole business of function theory on Riemann surfaces this became the main issue (for example, see Carleson's proof of the first Mergelyan theorem [**C**]). Papers [**3**, **4**, **11**] are ever-expanding generalizations of the F. and M. Riesz theorem, [**17**] demonstrates that the Rudin-Carleson theorem is a functional-analytic corollary of the F. and M. Riesz theorem, and [**5**, **10**] also proceed along the same lines.

But in [**2**], this concentration of attention to annihilating measures produces in addition a very strong approximation theorem: let S and R be as above. For C a compact subset of S, let $A(C)$ be the algebra of continuous functions on C that are analytic at interior points of C. Then there is a finite number of linear conditions such that every function in $A(C)$ that satisfies these conditions is approximable by functions in R. This is a generalization of Mergelyan's approximation theorem on the plane.

A few years later we could see clearly that Theorem B was hiding in this paper after all: Lemma (6) (now known as "Bishop's splitting lemma") is dual to the vanishing of the first cohomology group of the structure sheaf (on S') associated to R. Vanishing of cohomology is essentially what one needs to show that what you can do locally can be done globally. Here, the splitting lemma is the tool that localizes the problem so that Mergelyan's theorem can be quoted. Later, the same kind of localization by dualizing Theorem B appears in the work of Douady [**D**] and Kerzman [**K**]. But by the mid '60s there was already much evidence that the dual approach was powerful in one complex variable, and it was hoped to be as productive in several variables. We even spent some idle time talking of homology of cosheaves, but this led nowhere.

Bishop alludes to this failure in [**19**] where he points out that "analogous methods are doomed to failure, as a modification of an example of Wermer [**W3**] will show." In this paper Bishop reconsiders S and R as above, but this time he drops the assumption of holomorphic convexity. In the papers [**W1**, **W2**] John Wermer showed how, by concentrating on explicit descriptions of representing measures, one could construct little pieces of Riemann surfaces throughout the spectrum of algebras of functions defined by analytic functions on a Riemann surface. These results lead one to believe that any algebra of analytic functions on a Riemann surface S can be realized as the algebra of functions on another Riemann surface S'. Of course now more is involved than making some identifications; great pieces of S' need to be constructed (for example, if R is the algebra of polynomials, and S is any open set in the plane, clearly S' consists of S together with all bounded components of the complement of S). In [**19**], Bishop shows that this can be done, using the analysis of representing measures much as did Wermer. In the words of Bishop: "The present paper carries the theory further in certain directions than did Wermer's work and develops some of the material more systematically. In particular a definitive theorem [that described above] about algebras of functions on Riemann surfaces is obtained."

A crucial tool in the construction of analytic structure is to find a function with discrete level sets (already utilized by Wermer) in the spectrum. Bishop hoped that this had a generalization to several variables, something like this: Now if R is an algebra of functions, and $F = (f^1, \ldots, f^k)$ is a k-tuple of functions in R, p a point in the spectrum S of R, and p is an isolated point of $F^{-1}(F(p))$, then there is a little piece of an analytic space in S through p. This is unfortunately untrue, but enough of it appears in [**21**], providing an alternative proof of Oka's pseudoconvexity theorem [**O**]. But more of this later.

2. Uniform algebras. The great strength of the work of Bishop and Wermer (and others) in complex analysis is that they were able to perceive problems from the point of view of functional analysis, allowing them to isolate the nuggets of hard analysis required to solve the problem. They were then able to attack the hard problems with their customary joy and abandon, and eventually crack them. The central approach was the Gelfand theory of Banach algebras: in particular, algebras of continuous complex-valued functions on a compact space closed in the topology of uniform convergence. "Since the usual terms for these algebras (sup. norm algebras or function algebras) are not euphonious, we shall call them *uniform algebras*." With this sentence Bishop in 1965 gave them the name that is now in common usage. The two main examples were $C(X)$ (all continuous functions on X) at one extreme, and at the other the closure of the algebra of holomorphic functions on a compact set in a Stein space. In those days every example we had was a pastiche of these two examples, and it was easy to believe that that was all there was. In [**12**], Bishop gave the first theorem attempting to verify this belief: if A is a uniform algebra on X, we can partition X according to the level sets of the real-valued functions in A, so that for every continuous function f on X, which on every such level set coincides with a function in A, we have that f is in A. Now it remains to show that if a uniform algebra has no real functions, then there is analyticity on most of X. It turns out that this is far from true, but some of the most telling theorems along this line are in Bishop's papers [**19**, **21**, **22**, **24**].

If A is a uniform algebra on X, and h is a complex homomorphism of A, it is a central theorem of the Gelfand theory that h is given by integration against a positive measure on X (such are called *representing measures*). One of the central themes of Bishop's work is to find representing measures supported on the smallest possible set, and having the maximum number of good properties. In [**7**] he showed that there is such a set (unique in the metric case) on which there is a representing measure for every complex homomorphism. In [**9**] in collaboration with Karel deLeeuw, the assumption that A is an algebra was rendered superfluous; the theorem was reformulated for linear spaces of functions and reproven in that context. Going a step further, the theorem appears as a special case of this assertion: for C any convex set in a topological vector space E, every point in C is the barycenter of a measure supported on the set of exposed points of C (a point is *exposed* if it is the intersection of C with a hyperplane). It wasn't even known when such exposed points exist; in fact Klee [**KL**] had given an example of a bounded closed convex set in a Banach space with no exposed points. However, he conjectured that if C had interior then such points exist. In [**20**], in collaboration with R. R. Phelps, the conjecture is verified, and (at least in the case that E has countable topology) they show that there are enough exposed points so that the assertion made above is true.

At the time this work was done Phelps and I shared an office in Campbell Hall. I'll never forget the joy in Bob's voice when they overcame the main obstacle and he proclaimed, "That Bishop is a genius." A sentiment all of us working with Bishop would share at one time or another.

Besides finding the best possible set to support representing measures, Bishop sought the best possible measures. Arens and Singer had observed quite a bit earlier that Jensen's inequality could be viewed as a fact about general uniform algebras rather than as a theorem of complex variables. In [**21**], Bishop showed that for every homomorphism of a uniform algebra, there is a representing measure for which Jensen's inequality holds. For most of us this seemed like just another nice step in the program of reducing the study of uniform algebras to function theory. But Errett had deeper purposes in mind; he saw that Jensen's inequality was the key to measuring the size of sets in terms of their negligibility from the point of view of function theory. This strange process, of first generalizing an abstract theorem from the special case, and then reapplying it in the abstract form to obtain new results in the special case, was one of Bishop's most telling techniques. We shall return to this use of Jensen measures.

3. Global theory of Stein spaces. Papers [**14**, **15**, **16**, **21**] constitute Errett Bishop's contribution to the analytic geometric study of analytic spaces. To Errett an analytic space (called "partially analytic") was a topological union of complex manifolds ordered by decreasing dimension, such that each was contained in the closure of its predecessor. A continuous function on such an object is "analytic" if it is in fact holomorphic on every one of the complex manifolds. There are many bizarre examples of such objects; enough to conclude that there can be no good geometry of them. But Errett required that there be many global analytic functions, at least enough to separate points. He intuitively understood that the existence of analytic functions forced this very simple-minded geometric object to have all the attributes of the analytic spaces as defined by Grauert and Remmert. Therefore he should be able to derive everything one knew of analytic spaces (and more!) by concentrating on the function theory and cleverly manipulating those global functions. In [**14**] he does exactly that, ending with a proof of Remmert's embedding theorem of Stein spaces, starting only from this very elementary description of (partially) analytic spaces.

The central tool of this work is the "special analytic polyhedron" whose ubiquitous existence is asserted in Theorem 3 of [**14**]. As alluded to at the beginning, this tool makes a Stein space (from the function-theoretic point of view) appear as little more than a fancy elaboration of the polydisk. The induction Oka employs to solve the Cousin problems already capitalizes on this conception, but here and in [**15**, **16**] it flowers into the great dispatcher of hard theorems. Once again the Cousin problems are solved, but now with no induction, only elementary functional analysis (inspired by the ideas in [**2**]) and special analytic polyhedra. Similarly, the Remmert-Stein theorem on closures of analytic spaces, and the blowing down theorem for holomorphically convex spaces are quickly proven. Much of Chapters V and VII of [**GR**] is derived from these papers. The nuggets of deep insight are everywhere dense. As an example, Theorem 8 of [**16**] states that a partially analytic space with enough analytic functions to separate points is an analytic space. The proof sets up an appeal to the Baire category theorem using the Noetherian property of local rings of analytic functions. When Errett first told me this argument, I was struck by how natural he thought it was to integrate algebra with an analysis in such an essential way. I came to realize that in Errett's mind all the pieces of mathematics fit together in a holistic way that was quite different from the way most people did science in those days.

4. Analytic structure. We have already alluded several times to the continuing search for analyticity in the spectrum of a uniform algebra. We recalled that Wermer, in the late '50s, had developed powerful methods in the one-dimensional case, and in **[19]** these techniques in the hands of Bishop were given new meaning. Meanwhile Gleason **[G]** had shown that if the kernel of a homomorphism of a uniform algebra is finitely generated, then a neighborhood of that homomorphism (as a point in the spectrum) has the structure of an analytic space. Contemplation of this proof led us to a natural and elegant construction of the envelope of holomorphy of a domain spread over C^n (such a domain is a complex manifold M of dimension n together with a map to C^n with nowhere vanishing Jacobian). The envolope of holomorphy is then just the spectrum $E(M)$ of the algebra A of functions holomorphic on M, and our observation was that by writing down the Taylor series of an analytic function, we could immediately deduce that $E(M)$ had the structure of a domain spread over C^n, with A appearing as the algebra of holomorphic functions on $E(M)$. Now came the hard part: show that $E(M)$ was a Stein manifold. Proofs of this fact had been given ten years before, and the central idea to those proofs was pseudoconvexity. In **[21]** Bishop was able to generalize to n dimensions just enough of the ideas of **[19]** to give a function-theoretic proof of this theorem.

[21] is full of beautiful and deep ideas, some of which have become standard tools in complex analysis, and others which have not yet been sufficiently tapped. At the time that was written, Errett could see how to easily prove the Remmert-Stein theorem on the removal of singularities of analytic sets using the concept of special analytic polyhedra. Together we further extended these ideas to prove also the Grauert-Remmert proper mapping theorem. Since these proofs were being incorporated into **[GR]**, Errett saw no need for independent publication. But he did see that these arguments, together with a skillful use of Jensen's inequality (guaranteed by the existence of Jensen measures, proven in **[21]**) could push the theorem on removable singularities much further. In **[23]** he does so, with this introduction: "We show that an analytic set A of pure dimension k defined in the complement of an analytic set B can always be continued through B, in case the $2k$-dimensional volume of A is finite or $\overline{A}\backslash B$ has $2k$-dimensional Hausdorff measure 0. The first of these results was conjectured by Stoll, and it can be applied to give a simple proof of Stoll's theorem that an analytic subset of C^n is algebraic if its volume of appropriate dimension doesn't grow too fast near infinity. Along the way we give simple proofs of the theorems of Radó and Remmert and Stein, and derive some interesting properties for representing measures in certain algebras of analytic functions. In the last section we introduce a general notion of capacity and use it to prove a very general extension of the theorem of Remmert and Stein." These ideas formed the basis of further work on these problems, which appeared in the monographs **[S, SI]**.

5. Real submanifolds of complex submanifolds. Although linear partial differential operators of first order with real coefficients are completely understood by the Frobenius theorem, there were no theorems about them once the coefficients were allowed to be complex. In 1956 Hans Lewy [**L1**] gave an example of such an operator L in three real variables such that the inhomogeneous equation $Lu = f$ had no solutions. In the process he showed that if u_1, u_2 are two independent solutions of the homogeneous equation, then every solution is a holomorphic function of u_1, u_2. In a subsequent paper [**L2**], Lewy generalized this observation to all such operators L with $[L, \overline{L}] \neq 0$, defined on an open neighborhood U of the origin in R^3, by showing that the functions u_1, u_2 realize U as a real hypersurface in C^2 lying on the boundary of a domain D such that every solution of $Lu = 0$ extends into D as a holomorphic function. In this way the subject of CR manifolds was born. The key to Lewy's argument was the observation that a moving complex hyperplane (suitably chosen) intersected U in a family of curves bounding disks, and the integrated (dual) version of the condition $Lu = 0$ said that u was, on each such curve, the boundary value of a function analytic in the corresponding disk. Bishop realized that this latter argument generalized to any CR-manifold in C^n, *so long as the appropriate family of curves bounding disks existed.* However, no way of wiggling hyperplanes or hypersurfaces would generate the desired curves. Ultimately Bishop realized that since a direct construction was not forthcoming, he would just have to *prove* that they existed. In [**25**] he did exactly that for k-dimensional CR-manifolds in C^n with $k > n$. There is a large volume of subsequent literature on these "Bishop's disks," always attempting to explain the phenomenon of analytic continuation of CR-functions.

In this paper Bishop recalls the observation made by Andy Browder that, for cohomological reasons, an orientable compact k-dimensional manifold M in C^k cannot be polynomially convex: this means that the algebra of CR-functions on M must extend somehow into a bigger set. His theorem on these analytic disks attempts to explain how. He goes on to say: "These problems seem to be very difficult. At least it is hard to prove global results. Therefore in this paper we consider primarily the local situation. Our only global result [has] to do with the exceptional points on a two sphere imbedded in C^2 ...." In other words, the only global result really is a local result using the existence of special points. Elsewhere [**COCA**, problems] Bishop observes that for the distinguished boundary B of the polydisk in C^2, there exist no local families of such analytic discs, although clearly there are large curves on B that bound analytic disks. He asks therefore to show that on any 2-torus in C^2 there eixst closed curves that bound Riemann surfaces in the ambient space. Errett worked very hard to solve this problem; in particular he tried to show that in any deformation of B, the analytic discs move along as well. Nothing came of these attempts, and ony recently has there been some progress on hulls of tori.

There is much of Errett's work that I have not covered: in particular his early work in operator theory, and finally the main work of his life in constructive mathematics. That I have not done so should not reflect on the significance of that work; only on my competence with it. The mathematical