BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 19, Number 2, October 1988

0273-0979/88 $1.00 + $.25 per page

Eigenvalues and s-numbers by Albrecht Pietsch. Akademische Verlagsgesellschaft Geest and Portig K.-G., Leipzig, and Cambridge University Press, Cambridge, New York and Melbourne, 1987 (Cambridge Studies in Advanced Mathematics, vol. 13) 360 pp., $59.50. ISBN 0-521-32532-3

From [**P1**]: "At the turn of this century, I. Fredholm created the determinant theory of integral operators. Subsequently, D. Hilbert developed the theory of bilinear forms in infinitely many unknowns. In 1918 F. Riesz published his famous paper on compact operators (vollstetige Transformationen) which was based on these ideas. In particular, he proved that such operators have an at most countable set of eigenvalues which, arranged in a sequence, tend to zero. *Nothing was said about the rate of this convergence* (emphasis the reviewer's)".

"On the other hand, I. Schur had already observed in 1909 that the eigenvalue sequence of an integral operator induced by a continuous kernel is square summable. This fact indicates that something gets lost within the framework of Riesz theory. The following problem therefore arises:

Find conditions on the operator T that guarantee that the eigenvalue sequence $(\lambda_n(T))$ belongs to a certain subset of c_0, such as l_r with $0 < r < \infty$."

(Here, c_0 denotes the space of null sequences with the sup norm and l_r the space of r-summable sequences with norm $\|(a_n)\|_r = (\sum_{n=1}^{\infty} |a_n|^r)^{1/r}$).

This is the basic idea of the book under discussion.

Paraphrased from [**R**]: Cauchy proved that if $f(\lambda) = \lambda^n - t_1\lambda^{n-1} \cdots \pm t_n = 0$ is the characteristic equation of an $n \times n$ matrix A then t_i is the sum of all the principal i-rowed minors of A. Thus the first coefficient t_1 is given by

$$t_1 = a_{11} + a_{22} + \cdots + a_{nn}$$

and is called the *trace* of A. Of course t_1 is also the sum of the roots of $f(\lambda)$, i.e., the sum of the eigenvalues of A (counting multiplicities). Thus, in the matrix setting, trace is linear (sum of diagonal elements) and is also the sum of the eigenvalues. All of this extends naturally to finite rank operators on paired linear spaces $[E, E']$: Every finite rank operator $T\colon E \to E$ can be written

$$Tx = \sum_{i=1}^{n} f_i(x)x_i, \qquad f_i \in E',\ x_i, x \in E.$$

The number

$$\phi\text{-trace } T = \sum_{i=1}^{n} f_i(x_i),$$

the "functional trace," is independent of the finite representation.

Letting $F(E)$ denote the finite rank operators on E, ϕ-trace as defined above is a linear functional on $F(E)$. Naturally one seeks topologies on F so that "ϕ-trace" extends to a continuous linear functional on the closure $\overline{F}$ of F. The other natural questions, of course, are

(i) Does the "spectral trace", σ-trace $T = \sum \lambda_i(T)$, $(\lambda_i(T))$ the eigenvalues of T, make sense?

(ii) If σ-trace exists is it linear? Is it continuous with respect to the given topology on $\overline{F}$?

(iii) If ϕ-trace and σ-trace both exist are they equal?

Finding (various) answers to these questions is the subject of this impressive book.

The natural starting place is Hilbert space. In 1936 Murray and von Neumann **[M]** isolated what they called the "trace class" operators on Hilbert space (more or less) as follows:

For $T \in L(H)$, the bounded linear operators on a Hilbert space H, let

$$\alpha_n(T) = \lambda_n(\sqrt{TT^*}),$$

where, as usual, $\lambda_n(\cdot)$ are eigenvalues and $\sqrt{TT^*}$ denotes the unique positive square root of TT^*. The $\alpha_n(\cdot)$ are called the *singular numbers* of the operator (the s-numbers of the title). The *trace class operators* on H, $S_1(H)$, are those operators

$$S_1(H) = \left\{ T \in L(H) \colon \sigma_1(T) = \sum_{n=1}^{\infty} \alpha_n(T) < +\infty \right\}.$$

Letting

$$S_2(H) = \left\{ T \in L(H) \colon \sigma^2(T) = \left(\sum_{n=1}^{\infty} \alpha_n(T)^2 \right)^{1/2} < +\infty \right\},$$

we obtain the *Hilbert-Schmidt* operators.

It is readily seen that

$$S_1(H) = S_2(H) \circ S_2(H),$$

i.e., every trace-class operator is the composition of two Hilbert-Schmidt operators (the original definition of trace-class) and conversely. On the other hand, these operators are precisely those admitting a representation

$$\text{(N)} \qquad T(x) = \sum_{n=1}^{\infty} f_n(x), x_n; \qquad \sum_{n=1}^{\infty} \|f_n\| \, \|x_n\| < +\infty.$$

It turns out that

$$\phi\text{-trace } T = \sum_{n=1}^{\infty} f_n(x_n)$$

is independent of the representation (N) and moreover,

$$|\phi\text{-trace } T| \leq \sigma_1(T) = \sigma_2 \circ \sigma_2(T),$$

where $\sigma_2 \circ \sigma_2(T) = \inf \sigma_2(A)\sigma_2(B)$. Here, the infimum is over all representations $T = AB$, A, B Hilbert-Schmidt operators.

The most fundamental relationship between $\lambda_n(T)$ and $\alpha_n(T)$ was discovered by H. Weyl **[W]** in 1949: For a compact operator T on Hilbert space

$$\text{(W)} \qquad \prod_{i=1}^{n} |\lambda_i(T)| \leq \prod_{i=1}^{n} \alpha_i(T).$$

Here the $(\lambda_i(T))$ are ordered by decreasing modulus and counting multiplicities. This result is now called the (multiplicative) Weyl inequality. It follows from (W) that for $0 < p < +\infty$

$$\sum_{n=1}^{\infty} |\lambda_n(T)|^p \leq \sum_{n=1}^{\infty} \alpha_n(T)^p.$$

So, for trace class operators, the eigenvalues $(\lambda_n(T))$ are absolutely summable and thus σ-trace T exists in this case. However, it wasn't until 1959 that Lidskii [**L**] filled in the gap

$$\phi\text{-trace } T = \sigma\text{-trace } T$$

for trace class operators on Hilbert space. We should mention that Grothendieck [**G2**] *said* it was true as early as 1955!

With the Lidskii result the Hilbert space picture became clear.

Now, as usual, the game is to move from this very favorable Hilbert space setting to that of Banach spaces.

The expression (N) makes sense in any Banach space and on arbitrary Banach spaces, Grothendieck [**G1**, **G2**] called such operators *nuclear*. (Ruston [**Ru**] introduced a similar concept.) Moreover, Grothendieck showed that

$$\sum_{n=1}^{\infty} |\lambda_n(T)|^2 < +\infty$$

is the best one can say for eigenvalues of nuclear operators on $C(K)$ or $L_1(\mu)$-spaces. He also showed that ϕ-trace T, $\sum_{n=1}^{\infty} f_n(x_n)$, is intimately connected with the metric approximation property. An example of Enflo [**E**] showed that not all Banach spaces have this property. I've omitted definitions and rearranged history a bit here in order to make the next statements.

Thus, the nuclear operators, while generalizing the Von-Neumann–Murray trace class to Banach spaces, turned out to be a poor generalization from the point of view of eigenvalue distribution and traces. Something else was needed for this direction!

In 1957 D. Eh. Allakhveidiev [**A**] in "an exotic journal" (quoted from [**P1**]) proved that in Hilbert space

$$\text{(A)} \qquad (\lambda_n(\sqrt{TT^*}) =) \quad \alpha_n(T) = \inf\{\|T - A\| \colon \operatorname{rank} A < n\}.$$

The expression on the right makes sense in any Banach space and led Pietsch [**P2**] to define the approximation numbers of an operator on a Banach space by expression (A) and to define for a Banach space X

$$S_p(X) = \left\{T \in L(X) \colon \sum \alpha_n(T)^p < +\infty\right\}.$$

Thus S_1 generalizes the trace class and S_2 generalizes the Hilbert-Schmidt operators to Banach spaces. We remark that for any Banach space X

$$S_1(X) \subset N(X)$$

but, unless X is a Hilbert space, the containment is proper.

Moreover, S_2 is a completely different generalization of Hilbert-Schmidt operators to Banach space from that given earlier by Grothendieck [**G2**]:

$T \in L(X,Y)$ is absolutely two summing (Grothendieck: semi-integral á gauche) provided there is a $K > 0$ such that for $x_1, \dots, x_n \in X$

$$\text{(2-S)} \qquad \left(\sum_{i=1}^{n} \|Tx_i\|^2\right)^{1/2} \leq K \sup_{\|f\|=1} \left(\sum_{i=1}^{n} |f(x_i)|^2\right)^{1/2}.$$

The class of these operators is denoted by $\Pi_2(X,Y)$. The smallest such K is the 2-summing norm $\pi_2(T)$. (Replacing 2 by p in (2-S) defines the class $\Pi_p(X,Y)$).

Denote by $\Pi_2 \circ \Pi_2(X)$ those $T \in L(X)$ for which there is a Y (which may change with T) and $A \in \Pi_2(X,Y)$, $B \in \Pi_2(Y,X)$ with $T = BA$. Then $\Pi_2 \circ \Pi_2(X)$ also generalizes "trace class" to Banach spaces. Again we emphasize that for Banach spaces S_1, $\Pi_2 \circ \Pi_2$ and N are in general different animals. But, remember, in the words of Barry Simon [**S**] "operator theory on Banach spaces is likely to be a zoo!"

König [**Kö2**] proved that for $T \in S_1(X)$ or $T \in \Pi_2 \circ \Pi_2(X)$

$$\phi\text{-trace } T = \sigma\text{-trace } T = \sum_{n=1}^{\infty} \lambda_n(T).$$

These results and many others are presented in this book. Aside from the questions alluded to above, the book under review addresses the following general problem (slightly rephrased from earlier):

"*Given a class of operators* A (technical term: *Ideal*) *on a Banach space* X *what is the eigenvalue distribution of operators in* A? Generally the optimal Lorentz space $l_{p,q}$ is found, i.e., $T \in A(X)$ (operators with some specified property) implies $(\lambda_n(T)) \in l_{p,q}$ and the exponents p,q are optimal for the class A. (The Lorentz space $l_{p,q}$ consists of all (complex) sequences (a_n) such that $(n^{1/p-1/q}a_n) \in l_q$. An appropriate sup is required if $q = +\infty$.)

THE TOOLS. Some facts used are standard: e.g., the structure of the underlying space ($L_p(\mu)$-spaces, Sobolev or Besov spaces) and the structure of the operator (classical integral operators on L_p given by a kernel K with various smoothness properties). A special case of this is convolution operators on L_p. In this important case the eigenvalues of a convolution operator $T_f(g) = f * g$ are just the Fourier coefficients of f. Standard interpolation methods are used throughout. Also the theorem on tensor stability on Holub [**H1, 2**] ($T \in \Pi_p(X_1, Y_1)$, $S \in \Pi_p(X_2, Y_2)$ implies

$$T \otimes S \in \Pi_p(X_1 \otimes_\varepsilon X_2, Y_1 \otimes_\varepsilon Y_2)$$

and $\pi_p(T \otimes S) \leq \pi_p(T)\pi_p(S)$) plays an important role. (This symbolism denotes the injective completion of $X \otimes Y$.)

Other important tools are the standard inequalities of Banach spaces, e.g., Hölder, Kinchine, etc. More recently developed tools, essentially introduced by König [**Kö1, Kö2**] and Pietsch [**P2, P4**], are used extensively. The simplest idea used is that of *Related Operator* [**P4**]: If $T = BA$, $A \in L(X,Y)$, $B \in L(Y,X)$, then A and B are related operators. (That is, every pair (A,B) with A in $L(X,Y)$ and B in (Y,X) is a related pair!) The utility of this notion is that for related operators A, B, AB and BA have the same nonzero eigenvalues with the same multiplicities. In fact they have the same spectrum. Actually Sylvester has proved this result for matrices as early as 1883!

This notion of related operators, simple as it is, provides quick proofs, for example, of the Grothendieck result mentioned earlier: If $T \in N(X)$ then $\sum_{n=1}^{\infty} |\lambda_n(T)|^2 < +\infty$.

Probably the principal tool employed is the generalized Weyl inequality. Pietsch, in a series of papers, developed an axiomatic approach to s-numbers. The approximation numbers, $\alpha_n(T)$, are the largest (under coordinate-wise ordering as sequences) s-numbers on Banach spaces. There are (too) many of these s-number sequences so we will not give precise definitions. For the record, in [**P1**] *nine* such sequences together with generalizations are introduced. Luckily these nine s-number sequences coalesce on Hilbert space to the familiar $\lambda_n(\sqrt{TT^*})$. However, the Weyl numbers, discovered by Pietsch, are truly important. The nth Weyl number of $T \in L(X, Y)$ is defined by

$$x_n(T) = \sup\{\alpha_n(TA) \colon A \in L(l_2, X),\ \|A\| \leq 1\}.$$

Concerning these numbers Pietsch proved the following [**P3**]: Let T be a compact operator on a Banach space X. The eigenvalues of T satisfy

$$\prod_{i=1}^{n} |\lambda_i(T)| \leq (2e)^{n/2} \prod_{i=1}^{n} \dot{x}_j(T),$$

where the $(\lambda_i(T))$ are arranged by decreasing modulus and counting multiplicities and $(\dot{x}_i(T))$ denotes the doubled sequence

$$x_1(T), x_1(T), x_2(T), x_2(T), \ldots.$$

I had to explain to a seminar once that the inequality did not mean to square the $x_i(T)$; for example, if $n = 5$ we have

$$\prod_{i=1}^{5} |\lambda_i(T)| \leq (2e)^{5/2} x_1(T)^2 x_2(T)^2 x_3(T).$$

In particular this generalized Weyl inequality allows one to deduce the important fact that if $(x_n(T))$ is in the Lorentz space $l_{p,q}$ then so is $(\lambda_n(T))$.

To reiterate, this inequality (with variations) is the main tool of the book.

Another useful tool, due to König [**Kö1**, **Kö2**], is that of *Uniform Riesz type* 1: An operator T on X is a Riesz operator if $T - \lambda I$ has finite dimensional kernel and closed finite co-dimensional range for each nonzero complex λ. In particular, the spectrum of such operators consists of eigenvalues of finite multiplicity and has no limit points except (possibly) zero.

I can't put it off any longer: for arbitrary Banach spaces X, Y let $A(X, Y)$ be a subset of $L(X, Y)$ that is closed under addition, contains the finite rank operators, and is closed under composition, i.e., if $R \in L(X_0, X)$, $S \in L(Y, Y_0)$, $T \in A(X, Y)$ then $STR \in A(X_0, Y_0)$. Let α be a quasi-norm satisfying

$\alpha(a \otimes y) = \|a\|\ \|y\|$, where $a \otimes y$ is the rank one operator $x \to a(x)y$, $a \in X'$, $y \in Y$;

$\alpha(S + T) \leq K[\alpha(S) + \alpha(T)]$, $S, T \in A$; and

$\alpha(STR) \leq \|S\|\ \|R\| \alpha(T)$, for $T \in A(X, Y)$.

Then (A, α) is called an ideal of operators. An ideal (A, α) is of uniform Riesz type 1 if:

(a) $A(X)$ consists only of Riesz operators and there is a C (depending on X, α) such that $T \in A(X)$ implies $\sum_{n=1}^{\infty} |\lambda_n(T)| \leq C\alpha(T)$; and

(b) if T, $T_n \in A(X)$, $\varepsilon > 0$ then $\lim_{n\to\infty} \alpha(T - T_n) = 0$ implies there is an N such that $\sum_{j=N}^{\infty} |\lambda_j(T_n)| \leq \varepsilon$ for all n.

The importance of this idea is the following result of König [**Kö1**]: If (A, α) is an ideal of operators of uniform Riesz type 1 then the spectral sum $\sum_{n=1}^{\infty} \lambda_n(\cdot)$: $(A, \alpha) \to \mathbf{C}$ (complex numbers) is α-continuous for *all* Banach spaces X.

This result is crucial to the theory of trace ideals. The ideal (A, α) is a *trace ideal* provided for all Banach spaces X, Y the finite rank operators $F(X, Y)$ are α-dense in $A(X, Y)$ and the functional trace ϕ-trace$(\cdot)$: $F(X) \to \mathbf{C}$ is α-continuous.

MAIN RESULT. If (A, α) is a trace ideal of Uniform Riesz type 1, the Lidskii trace formula

$$\phi\text{-trace } T = \sigma\text{-trace } T = \sum_{n=1}^{\infty} \lambda_n(T)$$

holds for *all* Banach spaces X and all operators $T \in A(X)$. Thus (finally) the book under discussion seeks to *find trace ideals of uniform Riesz type* 1.

Important examples are $\Pi_2 \circ \Pi_2$ and S_1 mentioned earlier.

AXIOMS AND ALL THAT. Most of the above is in the mathematical mainstream. The author in his axiomatic way defines a (generalized) trace τ on an ideal (A, α):

$$\begin{array}{ll} \tau(a \otimes x) = a(x) & \text{for } a \in X',\ x \in X; \\ \tau(ST) = \tau(TS) & \text{for } T \in A(X, Y),\ S \in L(Y, X); \\ \tau(S + T) = \tau(S) + \tau(T) & \text{for } S, T \in A(X); \text{ and} \\ \tau(\lambda T) = \lambda\tau(T) & \text{for } T \in A(X),\ \lambda \in \mathbf{C}. \end{array}$$

We mention here that there is a wild example due to Kalton [**K**] (remember we're in a zoo) of an operator ideal (A, α) supporting many different continuous traces!

THE COMPETITION. All in all, this is a wonderful book. It is definitely *not* a sequel to the author's "Operator Ideals." Although the latter book has been made a principal reference to [**P1**] it is really not necessary and overlap has been kept to a minimum. "Operator Ideals" uses excessive notation and hence, is difficult to read. This is not the case here. Notation can be a problem: I don't like, e.g., "$L_{p,u}^{(a)} \circ H \circ (\Pi_2)_{q,v}^{(a)} = (\Pi_2)_{r,w}^{(a)}$" (p. 106) or "$C_{\text{op}}^{f}$" (p. 278). Although the meanings are clear in context, a few words wouldn't hurt. For the most part the notation is, however, straightforward and highly readable. This book is definitely not in competition with "Operator Ideals."

However, it is definitely in friendly competition with Hermann König's "Eigenvalue Distribution of Compact Operators" [**Kö1**]. Here we have a real mutual admiration society. Pietsch refers to König $f(n)$-times and König refers to Pietsch $g(n)$-times, the functions are strictly increasing, and $\lim_{n\to\infty} f(n)/g(n) = 1$.

The intersection of these books is quite large but the emphasis is totally different.

The historical survey is Pietsch's book is worth the price of the book. Covering the period 1646–1986 this history is scattered with often amusing quotes and anecdotes of the mathematicians involved. This survey should be of interest to *all* mathematicians and not limited to operator theorists.

The bibliography of [**P1**] is also an important document. Listing 82 books and monographs and 366 papers covering the spectrum from D'Alembert (1743) to Pietsch (1986).

What more can I say. König's book is fit for a king—Pietsch's book is peachy!

FROM THE PREFACE OF [**P1**]: "Many classical treatises on integral equations ... were published either in Cambridge or Leipzig. Therefore I regard it as a good omen that this monograph will be (now is: Reviewer) jointly issued by publishing houses in these cities. Moreover, in my opinion such undertakings are valuable contributions to scientists and editors to the realization of a peaceful coexistence of mankind."

Right on Professor Pietsch!

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 19, Number 2, October 1988

0273-0979/88 $1.00 + $.25 per page

Selected papers of Errett Bishop, edited by John Wermer. World Scientific Publishing, Singapore and Philadelphia, xxiv + 414 pp., $56.00. ISBN 9971-50-127-9

In the summer of 1959 I visited the Mathematics Department in Berkeley for the first time. There was a summer-long seminar in functional analysis that year, and practically everyone working in Banach algebras and associated topics spent some time there. I met Errett Bishop at one of the post-seminar teas, and we talked of problems in several complex variables related to function algebras. He asked me whether or not I knew if any analytic polyhedron in a complex manifold of dimension n could be approximated by one defined by only n functions. I hadn't a clue, and I asked him why he supposed such a thing should be true. His answer: "Well, for a projective variety of dimension n, it is true that almost every projection to P^n is of degree n, and that's really the same thing for this special case." I was unable to see the connection, and, being a brash upstart fresh out of MIT, I assumed that there wasn't one. We passed on to a discussion of minimal boundaries; that spring I had studied his paper on this subject and was very impressed by the appearance of "hard" analysis in what I thought was a subject in "soft" analysis. I wanted to calculate the minimal boundary of analytic polyhedra. Errett instantly knew where I was stuck, and suggested that I needed to understand better how to cut down representing measures using peak sets.

Several months later, while writing up a set of notes on analytic spaces, I finally understood Bishop's connection between generic projections of projective varieties and (what later became known as) special analytic polyhedra. I discovered how easy it was to extend known theorems for the polydisc to analytic covers of the polydisc, and that his idea, if true, amounted to the assertion that any Stein space could be approximated by analytic covers of polydiscs! What a potent tool! All theorems proven locally for analytic spaces by means of the parametrization theorem could now be proven for arbitrarily large domains on Stein analytic spaces. I called him to tell him about my discovery. He seemed pleased and interested and added this: "Furthermore, if I can see how to convert an almost proper map to a proper one, all these theorems will extend to the whole Stein space, and furthermore, provide an embedding into C^n." "You mean you can prove Remmert's theorem?" I asked. He replied that he thought so.

In that way began one of the most important mathematical friendships of my career. For most of the following two years, Errett Bishop was a member of the Institute for Advanced Study, and I was an assistant professor at Princeton