compactified moduli space, spaces of Kleinian groups, and applications to topology, geometry, and physics, in addition to some of the topics discussed in the last paragraph) is a sign of the vitality of the subject.

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Differential systems and isometric embeddings, by Phillip A. Griffiths and Gary R. Jensen. Annals of Mathematics Studies, vol. 114, Princeton University Press, Princeton, N. J., 1987, xii+225 pp., $35.00 (cloth). $15.00 (paper). ISBN 0-691-08429-7

A submanifold of any Euclidean space has an induced metric structure, and it is natural to wonder about the converse: Can a given Riemannian structure on a manifold M^n be induced by an embedding of M into some Euclidean space E^N? The search for local and global isometric embeddings has been fruitful for mathematics. However the three major advances in the local theory have been relatively inaccessible even to many workers in the field.

In terms of partial differential equations, the local embedding problem reduces to solving

$$(1) \qquad \sum_{\nu=1}^{N} \frac{\partial u^\nu}{\partial x_i} \frac{\partial u^\nu}{\partial x_j} = g_{ij},$$

where (g_{ij}) is a positive definite, symmetric $n \times n$ matrix.

When $n = 1$ such a solution clearly exists, and one may take $N = 1$. For $n = 2$ there are many special results, some of which are old [**We**] and others quite recent [**Li**]. The basic question is still open: Can every two-dimensional C^∞ Riemannian manifold be locally isometrically embedded in E^3?

Notice that (1) becomes determined, in the sense that the number of equations equals the number of unknowns, when $N = \frac{1}{2}n(n+1)$. Most of our discussion will be for this dimension. Also notice that not even the case where g_{ij} is real analytic is immediate. The difficulty, in classic PDE terms, is that every hypersurface is characteristic for (1). So the Cauchy-Kowalewski theorem cannot be invoked at once. One approach, first used by Janet, is to differentiate (1) to obtain a second-order equation such that $\{x_n = 0\}$ gives a noncharacteristic hypersurface. The Cauchy-Kowalewski theorem may then be applied for any analytic initial data. Correct choice of this initial data, which means solving the isometric embedding problem in one lower dimension, leads to a solution of (1).

Janet's proof was complete only in the case $n = 2$. As he noted, there remained in higher dimensions an algebraic difficulty to be resolved. This was done by Burstin [**Bu**]. A somewhat different proof, based on [**Ja1**], can be found in Spivak's *Differential geometry* [**Sp**, pp. 216–225]. (In fact, as we shall point out, several sections of [**Sp**] can be used to provide background for a study of the book under review.)

A year after Janet's paper, Cartan published a proof based on his theory of differential forms. Nowadays, we think of his proof as an application of the Cartan-Kaehler theorem (although in fact he used earlier results, see below). Later we state the Cartan-Kaehler theorem, but its statement never reveals its basic nature as a methodical approach to real analytic (C^ω) geometric problems to which the Cauchy-Kowalewski theorem does not immediately apply. So the proofs of Cartan and Janet have something in common.

The proof of Cartan is the more interesting and influential. However, it was probably not widely understood at the time. It might not be out of place here to quote two general remarks about Cartan's work.

"I must admit that I found the book, like most of Cartan's papers, hard reading." ([**Wy**], the book is *La theorie des groupes fini et continus...*, and the reference is due to [**CC**].)

"Cartan is the greatest differential geometer of the previous generation. Few have read his works, many pretend to have read them and every one agrees that every one should read them. I get shell-shock every time I try." [**Sp**, p. 611]

The next major advance did not come until the work of Nash in 1956 [**Na**]. Here was a paper far more challenging to understand than Cartan's! Undoubtedly, the majority of mathematicians who tried to read this paper were unable to do so. But as its techniques were understood and simplified, as in [**Sc**, **Mo**, **Gr**, **Se**, **Ho**, **Ha**], etc., it became an important result in nonlinear functional analysis. Nash's contribution was to handle the C^∞ case (even globally) by taking N large enough. So then the game became to understand the C^∞ case in the same dimension as the C^ω, i.e., to establish a C^∞ Cartan-Kaehler theorem. Here we come to the third major advance in the local isometric embedding problem. A series of papers by Griffiths and his students, Bryant, and others started to solve this C^∞ problem. Again, this advance in the local isometric embedding is marked by inaccessibility since it builds upon a deep understanding of Cartan's work and in parts relies on heavy algebraic geometry. So, by analogy with the advances of Cartan and Nash, we might expect a significant lapse of time before the techniques become widely understood. But this book is a start at breaking the analogy.

The book is an expanded version of the William H. Roever Lectures in Geometry given by Phillip Griffiths at Washington University. It lays out those parts of Cartan's theory used in recent work and outlines the needed generalizations. The study [**Br**] (which the reviewer has not yet seen) should complete and extend this introduction and together they should make the recent and ongoing work accessible to many more mathematicians.

The book starts with a brief discussion of the classical results and a statement of new results [**BBG**, **BGY**] such as:

a. A nondegenerate local isometric embedding is rigid if $N \leq 2n$ (and $n \geq 8$).
b. There is an open dense set of three-dimensional C^∞ metrics which can be locally embedded into E^6.

This is followed by elementary material on the structure equations of submanifolds of Euclidean space and then the basic object of study of this book—Pfaffian differential systems (PDS).

The theory of such systems has a naturality and elegance and also a long history, yet its importance is undoubtedly in its myriad of applications. A PDS is given locally on a manifold M by

$$(2)\qquad \begin{array}{ll} \text{(i)} & \theta^\alpha = 0, \qquad \alpha = 1, \dots, s. \\ \text{(ii)} & \omega^1 \wedge \cdots \wedge \omega^p \neq 0, \end{array}$$

where θ^α, ω^ρ are independent 1-forms and we seek a submanifold of M on which the restrictions of θ^α, ω^ρ satisfy (i) and (ii).

At the most superficial level, PDS contains a reformulation of PDE. Consider a nonlinear PDE

$$(3)\qquad F(x, y, u, u_x, u_y) = 0.$$

Let $p = u_x$ and $q = u_y$, so (x, y, u, p, q) is a point in $\mathbf{R}^5$. Let M be the set of such points satisfying

$$(4)\qquad F(x, y, u, p, q) = 0.$$

We assume that M is a manifold. We seek a submanifold of M on which

$$\theta' \equiv du - p\,dx - q\,dy = 0, \qquad dx \wedge dy \neq 0.$$

Such a submanifold is clearly given by a graph $(x, y,\ U(x, y),\ P(x, y),\ Q(x, y))$ with $P = U_x$ and $Q = U_y$. Thus by (4), the function $U(x, y)$ solves (3).

But PDS applies to problems which cannot be easily expressed as PDE and even for PDE itself provides tools which go beyond classical methods. Indeed, here might be the most important lesson to learn from this book. After all, the real analytic result of Cartan was first essentially done by Janet in the framework of differential equations. So, arguably, there was no need for Cartan's more sophisticated approach. But the algebraic difficulties in handling the C^∞ case when $N \leq n(n+1)/2$ seem to preclude a study of (1) using classical PDE and dictate this more intrinsic approach.

Now let us return to the PDS (2). Of course $s + p \leq \dim M$. The simplest case is when $s + p = \dim M$. Then (2) is called a Frobenius system and it is well known that a necessary and sufficient condition for solvability is

$$d\theta^\alpha = 0 \qquad \mathrm{mod}\,\{\theta^1, \dots, \theta^s\}.$$

(Some mathematicians know this better in the dual form: The r-plane distribution $\{X_1, \ldots, X_r\}$ admits a foliation by integral submanifolds if and only if $[X_j, X_h] \in \{X_1, \ldots, X_r\}$.) That is, the system must be "involutive." The definition of involutive when $s + p < \dim M$ is more complicated. Indeed, to quote from Griffiths and Jensen (p. 32):

"Although of considerable formal elegance and undeniable importance, the theory of differential systems is plagued by both a plethora of basic concepts and definitions, and at least some confusion concerning the basic notion of prolongation and involution. We will attempt to wade through a little of this without unduly aggravating the existing situation."

Prolongation is the process of adding to a system those equations that may be derived from the original equations by differentiation. Involution is the state of grace when (2) has no "hidden" compatibility conditions. The Cartan-Kuranishi theorem states that any PDS achieves involution after a sufficient number of prolongations and the Cartan-Kaehler theorem asserts that if a real analytic PDS is involutive then it has local integral manifolds. All of this is explained with examples and without proofs. One of the strengths of this book is that there is a well-thought-out intermingling of the general PDS theory and the isometric embedding problem, of theory and examples, of proofs and indications. Results are sometimes stated under mildly restrictive hypotheses in order to have clean statements, and technical details are often deferred to [**Br**]. This makes for a crisp, attractive presentation. Readers desiring a more leisurely treatment of differential systems up to the Cartan-Kaehler Theorem might look at [**Sp**, Addendum 1 to Chapter 10].

The presentation of classical PDS theory is soon specialized to the context of a newly introduced class, the quasilinear PDS, which is defined by: $d\theta^\alpha \equiv 0 \mod \{\theta, \omega\}$. The authors point out that many parts of the theory simplify for this class. In particular the authors outline a proof of Cartan's test for involutivity in this context.

It is now time to return to the isometric embedding problem. Following Cartan we start with a Riemannian manifold $\overline{X}^n$ and a Euclidean space E^{n+r} and we let $M = \mathscr{F}(\overline{X}) \times \mathscr{F}(E)$ be the product of the orthonormal coframe bundles. On $\mathscr{F}(\overline{X})$ we have the well-defined forms (obtained by lifting the forms which give each point of $\mathscr{F}(\overline{X}))\bar{\omega}^1, \ldots, \bar{\omega}^n$ and on $\mathscr{F}(E)$ the forms $\omega^1, \ldots, \omega^{n+r}$. An isometric embedding of $\overline{X}$ into E corresponds to the PDS,

$$\begin{gathered} \omega^i - \bar{\omega}^i = 0, \qquad i = 1, \ldots, n, \\ \omega^\mu = 0, \qquad n+1 \le \mu \le n+r, \\ \bigwedge_i \omega^i \bigwedge_{i<j} \omega^i_j \bigwedge_{\mu<\nu} \omega^\mu_\nu \neq 0, \end{gathered} \tag{5}$$

where

$$d\omega^i = -\sum_j \omega^i_j \omega^j \quad \text{and} \quad d\omega^\mu = -\sum_{a=1}^{n+r} \omega^\mu_\alpha \omega^\alpha.$$

So $\omega^1, \ldots, \omega^{n+r}$ becomes an adapted frame along the image of $\overline{X}$ in E and the embedding is isometric. From (5) we derive

$$\omega^i - \bar{\omega}^i = 0, \quad \omega^\mu = 0, \quad \omega^i_j - \bar{\omega}^i_j = 0. \tag{6}$$

Cartan showed that this system is in involution. It was known that such systems had solutions. (Cartan refers to [**Go**].) For details see, e.g., [**Sp**, Addendum to Chapter 11].

Note that (6) is a prolongation of (5). Griffiths and his co-workers prefer to take as the prolonged system (6) plus the equation

$$\omega_i^\mu - h_{ij}^\mu \omega^i = 0,$$

where (h_{ij}^μ) is a potential 2nd fundamental form and to take M as a certain submanifold of $\mathscr{F}(x) \times \mathscr{F}(E) \times \mathbf{R}^k$ where the variable in $\mathbf{R}^k$ is h_{ij}^μ. The reason for this preference is that this new system is quasilinear whereas (6) is not. It is shown that this new system is involutive for $n + r \leq n(n+1)/2$. The real analytic embedding theorem now follows.

Now we come to the heart of the recent work—the characteristic variety. Its usefulness leads the authors to state:

"Certainly, in practice when given a P.D.S. arising from a geometric problem the first thing one now does is to determine its characteristic variety."

It is shown that the characteristic variety for the isometric embedding problem is empty in low enough co-dimension and is "hyperbolic" in the determined three-dimensional case. This then leads, via results of Yang [**Ya**] and Berger, et al. [**BBG**] to the new results for the isometric embedding problem stated above.

The PDS for isometric embeddings is further investigated when the Riemannian manifold is a "space form," i.e., has constant sectional curvature. And then in the last chapter a different problem is investigated, the embedding problem for Cauchy-Riemann (CR) structures. Just as a Riemannian structure on a manifold is the abstraction of the structure induced on submanifolds of Euclidean space, a CR structure is the abstraction of the structure induced on submanifolds of complex space. And just as the natural question for an abstract Riemannian structure is whether it can be realized as a submanifold of Euclidean space, the natural question for an abstract CR structure is whether it can be realized as a submanifold of complex space. We restrict ourselves to real hypersurfaces X^{2n+1} in $\mathbf{C}^{n+1}$. Let V be the bundle of complex vectors which are tangent to X^{2n+1} and which are in the linear span of $\{\partial/\partial\bar{z}_1, \ldots, \partial/\partial\bar{z}_{n+1}\}$. Then $V \cap \overline{V} = \{0\}$ and $[V, V] \subset V$. So an abstract CR structure (of hypersurface type) is a manifold X^{2n+1} and a sub-bundle V of the complexified tangent bundle of X that has the properties $V \cap \overline{V} = \{0\}$ and $[V, V] \subset V$. It is not difficult to see that this structure is realizable as a real hypersurface in $\mathbf{C}^{n+1}$ provided there are functions h_a, $a = 0, \ldots, n$ which satisfy

$$Lh_a = 0 \quad \text{for all complex vector fields } L \text{ in } V$$

and

$$dh_0 \wedge \cdots \wedge dh_n \neq 0.$$

That is, the sub-bundle V should be homogeneously solvable.

(ADVERTISEMENT: An elementary survey of homogeneous solvability can be found in the reviewer's lecture notes [**Ja2**].)

Proceeding analogously to the isometric embedding problem, the authors set up a PDS for the realization of CR structures. They again compute the characteristic variety and as a consequence solve the real analytic problem.

As the authors point out, the C^∞ embedding problem is much more subtle. First of all, increasing N does not change the nature of the problem. More importantly in dimension 3 (so $n = 1$) there are counterexamples [**Ni**] and in fact the counterexamples are dense [**JT1**]. In dimensions 7 and higher local embeddings always exist provided the structure is "strictly pseudo-convex." This deep result is originally due to Kuranishi [**Ku**] for dimension 9 and higher and has been improved in [**Ak**] to include dimension 7. (Recall that our discussion of CR structures is restricted to those of hypersurface type so there is no dimension 8.) There are again counterexamples if strict pseudo-convexity is dropped [**Ea, JT2**].

The C^ω result itself is easy to prove directly without mentioning PDS for it follows, without much fuss and even only assuming analyticity in one variable, from the Frobenius theorem for vector fields. So the purpose of this chapter is not the result per se but to illustrate the general theory. And a more subtle purpose can be read into the chapter. The characteristic variety was the key to the new understanding of the isometric embedding problem and led to C^∞ results. But for CR structures the characteristic variety does not contain enough information. In particular it does not reflect the property of strict pseudo-convexity. Just as the characteristic variety led to the resolution of the isometric embedding problem, one might hope to now find a "golden bullet" for the CR embedding problem.

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Representations of rings over skew fields, by A. H. Schofield. London Mathematical Society Lecture Note Series, vol. 92, Cambridge University Press, Cambridge, London, New York, New Rochelle, Melbourne, Sydney, 1985, xii+223 pp., $27.95. ISBN 0-521-27853-8

Let k be a commutative field. For an associative k-algebra R with unit, a *representation* of R is a k-algebra homomorphism from R to a matrix ring $M_n(S)$ where S is also a k-algebra. To be precise, this is an n-dimensional representation of R over S. In the historical beginnings of this notion, S was assumed to be k, but it was soon seen to be useful to allow S to be an extension of k, chosen perhaps to decompose the representation. When R is finite k-dimensional and semisimple (the word semiprimitive is also used) the Wedderburn classification theorem states that R is a direct product of simple Artinian rings, i.e., those of the form $M_n(D)$ for D a skew field (division algebra) over k. Thus it is worthwhile to allow S to be a skew field in the definition of representation. While in the above case D will be finite k-dimensional, for infinite-dimensional R we may need to allow the most general notion of representation over any skew field.

Classically, representations $R \to M_n(D)$ have been considered equivalent if they are conjugate, i.e., if there is a conjugation (inner) isomorphism of $M_n(D)$ which carries one representation to the other. This is not sufficient in the new generality: not only might n and D be different for the two representations, but also the algebra $M_n(D)$ may have other isomorphisms. This is even true for $n = 1$, so we examine that case.