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Finally, for a demonstration of how to weave a spell with geometry and analysis on hyperbolic space, I recommend the beautiful article by F. Apéry in Gazette des Mathematiciens (1982), pp. 57–86, entitled *La baderne d'Apollonius*, which also provides some illustrations of the wide opportunities for computer graphics in this field, beyond the current obsession with fractal curves.

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Teichmüller theory and quadratic differentials, by Frederick P. Gardiner. John Wiley and Sons, New York, Chichester, Brisbane, Toronto, Singapore, 1987, xvii+236 pp., $46.95. ISBN 0-471-84539-6

The theory of moduli of Riemann surfaces occupies a central role in modern mathematics. Its origins lie in the classical theory developed in the nineteenth century, and it has attracted the attention of many of the outstanding mathematicians of the twentieth century, including Poincaré and Hilbert at the beginning of the century, Ahlfors and Bers during most of the middle half of this century, and Thurston and Sullivan at the present time. The subject is rich with deep general theories and full of interesting special cases. It has a technology of its own, but borrows extensively from other disciplines (topology, algebraic geometry, several complex variables) and has applications to diverse fields (partial differential equations, minimal surfaces, particle physics).

One of the main objects in the theory is the Teichmüller space $T(p,n)$ whose points are all the "marked" compact Riemann surfaces of genus p with n punctures or distinguished points. To avoid the elementary and easy to handle cases, one assumes that the surface has negative Euler characteristic; that is, both p and n are nonnegative integers with $2p-2+n > 0$. By a marking on a surface, we mean a choice of a basis for the fundamental group of the surface. It is an important observation that the space of marked surfaces is easier to study than the more natural object $R(p,n)$ consisting of conformal equivalence classes of compact surfaces of finite type (p, n), that is, the space of all conformal equivalence classes of Riemann surfaces of genus p with n punctures. The space $T(p, n)$ appears implicitly in the early continuity arguments of Klein and Poincaré. It has been constructed as a real manifold of dimension $6p-6+2n$ by Fricke, who proved that it was a cell (the fact that we do not call these spaces after Fricke is indeed ironic but not the greatest irony in the subject), and by Fenchel-Nielsen (in a mostly unavailable manuscript). By the way, many of the early authors treat only the case of closed surfaces (that is, $n = 0$). This is the most interesting case, and the introduction of punctures only occasionally involves fundamentally new ideas or approaches. The modern theory really begins with Teichmüller who introduced a complete metric on $T(p, n)$ that bears his name (1939). The metric brought quasiconformal mappings to the field in a very dramatic way. First, it introduced an important extremal problem: among all orientation-preserving topological mappings in a given homotopy class between two Riemann surfaces of finite type, find the one that is closest to conformal. Teichmüller's solution to the problem, while hard to follow because his sketchy arguments had numerous gaps (almost requiring a kind of faith to believe in their correctness), was brilliant. His solution is expressed in terms of holomorphic quadratic differentials on the surfaces. Second, it gave a new proof of the fact that $T(p, n)$ is a cell. Third, it showed the power of quasiconformal mappings and their intimate connections with classical complex analysis. Almost all the subsequent results obtained in Teichmüller theory use quasiconformal mappings in very basic ways. Fourth, it motivated Ahlfors (who gave a good flexible definition for quasiconformality) and Bers (who introduced quasi-Fuchsian groups) both to establish a solid foundation for Teichmüller's claims and to go much further in developing the theory.

It is useful at this point to inform the reader about two definitions. Quasiconformal mappings are natural generalizations of conformal mappings. An orientation-preserving homeomorphism is quasiconformal if it is conformal with respect to some Riemannian metrics of uniformly bounded eccentricity (if it maps infinitesimal circles onto infinitesimal ellipses with uniformly bounded ratios of major to minor axes). A quadratic differential is a section of the square of the canonical line bundle (an expression $f(z)\, dz^2$ that is invariant under changes in the local coordinate z).

Ahlfors (1960) proved that the Teichmüller space has a natural complex structure. His proof uses periods of abelian differentials of the first kind. This theorem is not unexpected; after all, compact Riemann surfaces are algebraic curves and hence should form a complex space of some kind. However, the original argument is difficult, quite clever, and was not easy to establish. Further, the space of moduli of Riemann surfaces is not a (complex) manifold. Bers (1966) embedded the Teichmüller space into the (finite-dimensional) vector space of integrable holomorphic quadratic differentials on a fixed Riemann surface, as a bounded domain of holomorphy. Bers uses univalent functions and Schwarzian derivatives. His methods relate Teichmüller theory to the classical theory of schlicht functions and mark a new appearance for quadratic differentials in the theory of deformations of Riemann surfaces. Quadratic differentials also appear as tangent and cotangent vectors to

Teichmüller space and in closely related topological subjects such as measured foliations on Riemann surfaces. It is impossible to study Teichmüller spaces without both quasiconformality and quadratic differentials. Bers's results also introduce a natural complex boundary for Teichmüller space. The boundary points represent degenerate Riemann surfaces in some way. For example, Riemann surfaces with nodes appear naturally as points in the Bers boundary of $T(p,n)$. However, these points do not exhaust the entire Bers boundary. Also present are totally degenerate Kleinian groups (Riemann surfaces are intimately connected to various classes of Kleinian groups; however, this is a subject for another book), where the surface has totally disappeared but has left a shadow or memory. This Bers boundary is still not well understood and poses many challenges to today's mathematicians. There is a natural group, the modular group $\mathrm{Mod}(p,n)$, that acts discretely as a group of complex analytic isometries of $T(p,n)$; the quotient $T(p,n)/\mathrm{Mod}(p,n)$ is the moduli space $R(p,n)$. It follows that this moduli space is hence a normal complex orbifold.

Every complex manifold has at least two natural metrics on it: the Carathéodory metric and the hyperbolic metric introduced by Kobayashi. Royden (1971) established the powerful result that the Teichmüller and Kobayashi metrics on $T(p,n)$ coincide. Thus the metric introduced by Teichmüller in a seemingly ad-hoc manner is quite natural and in fact recoverable from the natural complex structure. To complete this circle of ideas we mention one final result. Royden also shows that $\mathrm{Mod}(p,n)$ is essentially the full group of complex analytic automorphisms of $T(p,n)$, whenever this space is at least two dimensional.

Gardiner's book treats all the topics discussed above, including the infinite-dimensional theory whenever appropriate. The author assumes that the reader is familiar with basic graduate courses in analysis (both real and complex), algebra, and topology, and has some familiarity with Riemann surfaces and Fuchsian groups. He reviews these topics along with the necessary tools from quasiconformal mappings (the so-called "measurable Riemann mapping theorem"); some prior acquaintance with this last subject would, however, not hinder the reader. The author has a definite and singular point of view. His goal is to develop a few very basic and fundamental principles and then recover as much of the theory as possible as consequences of these basic principles. This objective leads naturally to an exclusion of certain important topics, for example, a detailed treatment of boundaries of $T(p,n)$ or of degeneration of surfaces or of fiber spaces over Teichmüller or moduli spaces and the associated interesting and important forgetful maps. Gardiner's approach has the great advantage of unifying a tremendous amount of mathematics. The two principles naturally involve quadratic differentials and quasiconformal mappings. He first establishes various uniqueness theorems that follow from the length-area principle of Grötzsch as amplified and reinforced in the pioneering work of Reich and Strebel and the more recent work of Marden and Strebel. From these rather general results on quadratic differentials, he establishes the basic inequalities of Reich and Strebel that are used in the study of extremal quasiconformal mappings, the uniqueness part of Teichmüller's theorem on extremal quasiconformal mappings, the sufficiency of the Hamilton-Krushkal condition for extremality, and certain uniqueness theorems for quadratic differentials that involve the metric induced by these differentials. He then discusses a second principle that describes the connection between families of trivial variations of the complex structure on a given surface and infinitesimally trivial (that is, trivial to first order) deformations. From this second motif, Gardiner establishes the existence part of Teichmüller's theorem as well as the necessity of the Hamilton-Krushkal condition (among other results). It is quite surprising that a large part of the theory of Teichmüller spaces is so intimately connected to just two important principles. The book closes with two chapters on quadratic differentials; the first of these develops the theory of Jenkins-Strebel differentials, the second develops parts of Thurston's theory of measured foliations.

The above is, of course, a very inadequate summary of the contents of this important book on Teichmüller theory. In establishing many of the fundamental results in the field, the author has had to deal with many topics not yet mentioned, for example, Poincaré series, approximation theorems for holomorphic functions, spaces of cusp forms, smoothness of metrics and cometrics on Teichmüller space. He has reproduced simplified versions of proofs appearing in some original papers, substituted new proofs for some theorems, and filled in the details to make some arguments more accessible. This is a scholarly work that nicely complements the existing books on Teichmüller theory. It is a worthy successor to the recently reprinted classic lecture notes of Ahlfors [**Ah**]. It covers many different parts of the theory than the books by Abikoff [**Abh**] and Krushkal [**K**]. It has a nonempty intersection with the more recent books of Strebel [**S**] and Lehto [**L**], as well as the forthcoming book of Nag [**N**]. The book by Strebel is an in-depth study of quadratic differentials, while the Lehto book is a more leisurely treatment of the foundations of Teichmüller theory and its strong relations to the theory of univalent functions.

Recently, I taught a semester course on Teichmüller theory. This book was extremely useful as background material. It was impossible to cover all the material in the book, and it was necessary both to augment a number of topics and to treat several subjects not covered in Gardiner's treatment. The natural "period map" from the Teichmüller space into the Siegel upper half-plane clearly belongs in any course on moduli. It shows that the complex structure given by the Bers embedding is the same one as the one introduced by Ahlfors, and hence is the natural complex structure. The infinitesimal form of this period map, known as the Rauch variational formula, introduces a name that is too often forgotten in works on Teichmüller theory. This formula shows that Noether's theorem is the infinitesimal form of Torelli's theorem, thus tying the modern theory once again to classical concepts. Also missing from Gardiner's book is the fact that (the Teichmüller curve over) Teichmüller space satisfies a universal mapping property, while Riemann space does not (because of the existence of surfaces with nontrivial automorphisms).

Gardiner's book is a well-written treatise that belongs in the library of every graduate student and scholar interested in complex analysis. It contains a wealth of information. Some topics are missing, of course. Together with the other books listed in the bibliography, this volume provides a good picture of the current state of a big portion of the theory of moduli of Riemann surfaces. The fact that more books on the subject are needed (treating, for example, compactified moduli space, spaces of Kleinian groups, and applications to topology, geometry, and physics, in addition to some of the topics discussed in the last paragraph) is a sign of the vitality of the subject.

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Differential systems and isometric embeddings, by Phillip A. Griffiths and Gary R. Jensen. Annals of Mathematics Studies, vol. 114, Princeton University Press, Princeton, N. J., 1987, xii+225 pp., $35.00 (cloth). $15.00 (paper). ISBN 0-691-08429-7

A submanifold of any Euclidean space has an induced metric structure, and it is natural to wonder about the converse: Can a given Riemannian structure on a manifold M^n be induced by an embedding of M into some Euclidean space E^N? The search for local and global isometric embeddings has been fruitful for mathematics. However the three major advances in the local theory have been relatively inaccessible even to many workers in the field.

In terms of partial differential equations, the local embedding problem reduces to solving

$$\sum_{\nu=1}^{N} \frac{\partial u^\nu}{\partial x_i}\frac{\partial u^\nu}{\partial x_j} = g_{ij}, \tag{1}$$

where (g_{ij}) is a positive definite, symmetric $n \times n$ matrix.

When $n = 1$ such a solution clearly exists, and one may take $N = 1$. For $n = 2$ there are many special results, some of which are old [**We**] and others quite recent [**Li**]. The basic question is still open: Can every two-dimensional C^∞ Riemannian manifold be locally isometrically embedded in E^3?

Notice that (1) becomes determined, in the sense that the number of equations equals the number of unknowns, when $N = \frac{1}{2}n(n+1)$. Most of our discussion will be for this dimension. Also notice that not even the case where g_{ij} is real analytic is immediate. The difficulty, in classic PDE terms,