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VARIETIES IN FINITE TRANSFORMATION GROUPS

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ABSTRACT. The equivariant cohomology ring of a G-space X defines a homogeneous affine variety. Quillen [**Q**] and W. Y. Hsiang [**Hs**] have determined the relation between such varieties and the family of isotropy subgroups as well as their fixed point sets when $\dim X < \infty$. In modular representation theory, J. Carlson [**Cj**] introduced cohomological support varieties and rank varieties (the latter depending on the group algebra) and explored their relationship. We define rank and support varieties for G-spaces and G-chain complexes and apply them to cohomological problems in transformation groups. As a corollary, a useful criterion for $\mathbf{Z}G$-projectivity of the reduced total homology of certain G-spaces is obtained, which improves the projectivity criteria of Rim [**R**], Chouinard [**Ch**], and Dade [**D**].

1. Introduction. Let G be a finite group. Assume in the sequel that all modules, including total homologies of G-spaces and G-chain complexes, are finitely generated. In a fundamental paper [**R**], D. S. Rim proved that a $\mathbf{Z}G$-module M is $\mathbf{Z}G$-projective if and only if $M|\mathbf{Z}G_P$ is $\mathbf{Z}G_P$-projective for all Sylow subgroups $G_P \subseteq G$. This theorem has had many applications to local-global questions in topology, algebra, and number theory. In his thesis [**CH**] Chouinard greatly improved Rim's theorem by proving that the $\mathbf{Z}G$-projectivity of M is detected by restriction to p-elementary abelian subgroups $E \subseteq G$, i.e. $E \cong (\mathbf{Z}/p\mathbf{Z})^n = \langle x_1, \ldots, x_n\rangle$. If M is $\mathbf{Z}$-free (a necessary condition for projectivity), it suffices to consider $k \otimes M$, where $k = \bar{\mathbf{F}}_p$ when restricting to E. In a deep and difficult paper [**D**], Dade provided the ultimate criterion: A kE-module M is kE-free if and only if for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in k^n$, the units $u_\alpha = 1 + \sum_{i=1}^n \alpha_i(x_i - 1)$ of kE act freely on M. Thus the projectivity question reduces to the restrictions to all p-order cyclic subgroups $\langle u_\alpha\rangle \subseteq kG$. Since $k = \bar{\mathbf{F}}_p$, all but finitely many are not subgroups of G. When the $\mathbf{Z}G$-module M arises as the homology of a G-space, we have a much simpler criterion which is a natural sequel to Dade's theorem.

THEOREM 1. *Let X be a connected paracompact G-space (possibly $\dim X = \infty$), and let $M = \bigoplus \bar{H}_i(X)$ with induced G-action. Assume that for each maximal $A \cong (\mathbf{Z}/p\mathbf{Z})^n \subseteq G$, the Serre spectral sequence of the Borel construction*

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$E_A \times_A X \to BA$ collapses. Then M is $\mathbf{Z}G$-projective if and only if $M|\mathbf{Z}C$ is $\mathbf{Z}C$-projective for all $C \subseteq G$ of prime order.

In applications, e.g. to G-surgery or equivariant homotopy theory, X arises as the cofiber of a highly connected G-map, and as such, the hypotheses are often verified.

COROLLARY. *Let $f: X \to Y$ be a G-map such that $H_i(f) = 0$ for $i \neq n$. Then $H_n(f)$ is $\mathbf{Z}G$-projective if and only if $H_n(f)|\mathbf{Z}C$ is $\mathbf{Z}C$-projective for all $C \subseteq G$ with $|C| =$ prime. ($H_*(f)$ is the homology of the cofiber.)*

COROLLARY. *Let X be a G-space such that nonequivariantly X is homotopy equivalent to a finite complex. Then X is G-equivariantly finitely dominated if and only if X is C-equivariantly finitely dominated for each $C \subset G$, $|C| =$ prime.*

The above theorem and its corollaries have local and modular versions also. E.g. $\mathbf{F}_pG$-projectivity is detected by restriction to cyclic subgroups of order p. In the sequel, $\mathscr{E}$ denotes the category of p-elementary abelian subgroups of G for a fixed prime p, where the morphisms are induced by inclusions and conjugations in G.

For a G-space X, Quillen defined the affine homogeneous variety $H_G(X)(k)$ to be the set of ring homomorphisms $H_G(X; \mathbf{F}_p) \to k$ with the Zariski topology [**Q**]. He showed that $H_G(X)(k)$ is completely determined by $\mathscr{E}(X) = \{E \in \mathscr{E}: X^E \neq \varnothing\}$ whenever $\dim X < \infty$, and that $H_G(X)(k) \cong \lim\operatorname{ind}_{E\in\mathscr{E}} H_E(X)(k)$. W. Y. Hsiang [**Hs**] showed that algebro-geometric considerations of $H_G(X; \mathbf{F}_p)$ lead to powerful fixed point theorems and the theory of topological weight systems. In a different direction, J. Carlson defined two varieties for a kE-module M, $E = (\mathbf{Z}/p\mathbf{Z})^n = \langle x_1, \ldots, x_n\rangle$. Namely, $V_E^r(M)$ (called the rank variety, inspired by Dade's theorem [**D**]) and $V_E(M)$ (called the cohomological support variety, inspired by Quillen's work [**Q**]). For $\alpha = (\alpha_1, \ldots, \alpha_n) \in k^n$, let $u_\alpha = 1 + \sum \alpha_i(x_i - 1)$ and define $V_E^r(M) = \{\alpha \in k^n: M|k\langle u_\alpha\rangle \text{ is not free}\} \cup \{0\}$. Then $V_E^r(M)$ is an affine homogeneous subvariety of k^n, and Dade's theorem translates into: M is kE-projective $\Leftrightarrow V_E^r(M) = 0$. On the cohomological side, consider the commutative graded ring $H_G = \bigoplus_{i\geq 0} \operatorname{Ext}_G^{2i}(k,k)$ and let $V_G(k)$ be the homogeneous affine variety of all ring homomorphisms $H_G \to k$ with the Zariski topology. The annihilating ideal $J(M) \subset H_G$ of the H_G-module $\operatorname{Ext}_G^*(M,M)$ defines the homogeneous subvariety $V_G(M)$. For $E = (\mathbf{Z}/p\mathbf{Z})^n$, there is a natural isomorphism $V_E(k) \cong V_E^r(k) = k^n$. Carlson showed that $V_E^r(M) \subseteq V_E(M)$, and Avrunin-Scott proved $V_E(M) = V_E^r(M)$ (proving a conjecture of Carlson) and $V_G(M) \cong \lim_{E\in\mathscr{E}} V_E^r(M)$ in analogy with Quillen's stratification theorem [**Cj**, **AS**, **Q**].

For a G-space X (similarly for a kG-complex) we define a cohomological support variety $V_G(X)$, and a rank variety $V_G^r(X)$ following Carlson's method. Since we will consider also $V_G(X, Y)$ for a pair (X, Y) of G-spaces, Quillen's definition is not used. Let $R = kG$. Two R-modules M_1 and M_2 are called (projectively) stably isomorphic if for some R-projective modules P and Q, $M_1 \oplus P \cong M_2 \oplus Q$. Let $\mathscr{S}(R)$ be the set of stable isomorphism classes. For

$n \in \mathbf{Z}$, define $\omega^n : \mathscr{S}(R) \to \mathscr{S}(R)$ by $0 \to \omega^1(M) \to P \to M \to 0$ and $0 \to M \to Q \to \omega^{-1}(M) \to 0$, where P is R-projective, Q is R-injective, and ω^n is defined by iteration. In $\mathscr{S}(R)$, the relation ($M_1 \sim M_2 \Leftrightarrow \omega^n(M_1) \cong \omega^n(M_2)$ for some n, $m \in \mathbf{Z}$) is an equivalence relation, called ω-stable equivalence. Let $\mathscr{S}^\omega(R) \equiv \mathscr{S}(R)/\sim$. These concepts have natural extensions to connected R-chain complexes with finitely many nonzero homology groups. The ω-stable equivalence classes of connected R-chain complexes form a set $\mathscr{S}_*^\omega(R)$. We define invariants for elements of $\mathscr{S}_*^\omega(R)$ to describe the asymptotic behavior of the hypercohomology (equivariant cohomology in the case of G-spaces) of the representative complexes. Let $\hat{\mathbf{H}}^*$ denote the Tate hypercohomology [**B**]. For $E \in \mathscr{E}$ and a kE-complex C_*, let $V_E^r(C_*) = \{\alpha \in k^n : \hat{\mathbf{H}}(\langle u_\alpha\rangle; C_*) \neq 0\} \cup \{0\}$, and $V_E(C_*)$ be the variety defined by the annihilating ideal of $\mathbf{H}^*(E; C_*)$ in the maximum spectrum of $H_E \equiv \bigoplus_{i\geq 0} H^{2i}(E;k)$. Let $V_G^r(C_*) = \varinjlim_{E\in\mathscr{E}} V_E^r(C_*)$ and $V_G(C_*) = \lim_{E\in\mathscr{E}} V_E(C_*)$. Then V_G^r and V_G are well defined on $\mathscr{S}_*^\omega(R)$, and $V_G([C_*]) = V_G^r([C_*])$. For a pair of G-spaces (X, Y), define $V_G(X, Y)$ and $V_G^r(X, Y)$ by taking an appropriate kG-chain complex $C_*(X, Y)$. The numerical invariants $\dim V_G^r(X, Y)$ (or $\dim V_G^r(C_*)$) are called the complexity of (X, Y) (respectively C_*) and

$$\dim V_G^r(X, Y) = \min\left\{n\colon \lim_{s\to\infty} \frac{\dim H_G^s(X, Y)}{s^n} = 0\right\}$$

(and similarly for $\dim V_G^r(C_*)$ using hypercohomology). This generalizes J. Alperin's notion of complexity of a kG-module M [**Aj**] and J. Carlson's interpretation of $\dim V_G^r(M)$ in terms of its complexity.

SKETCH OF PROOF OF THEOREM 1. The hypotheses on X and the ω-stability of the varieties lead to the equalities

$$V_G(\tilde{C}_*) = V_G^r(\tilde{C}_*) = V_G^r\left(\bigoplus_i \bar{H}_i(X; k)\right)$$

for each prime $p||G|$. A reinterpretation of geometric points of $V_G(\tilde{C}_*(X))$ using a theorem of Serre [**S**] (in the spirit of Quillen's description of subvarieties of $H_G(X)(k)$ in terms of prime ideals associated to $\mathscr{E}(X)$) leads to the desired projectivity criterion.

REMARKS. (1) The theorem is not true if the Serre spectral sequence does not collapse. The nonzero differentials give other interesting geometric invariants, in particular for infinite dimensional G-spaces.

(2) If $g \in G$ acts on X by self-homotopy equivalences (i.e. a homotopy G-action), then $H_*(X)$ becomes a $\mathbf{Z}G$-module again. One defines rank varieties for homotopy G-actions similarly, where they are considered as a natural algebraic substitute for the notion of isotropy subgroups [**Aa**].

(3) It is easy to construct nonprojective $\mathbf{Z}G$-modules whose restrictions to prime order subgroups are projective. Using Theorem 1, one provides counterexamples to Steenrod's problem for $G \supset (\mathbf{Z}/p\mathbf{Z})^2$ or $G \supset Q_8$, thus giving another proof of theorems of G. Carlsson [**Cg**] and P. Vogel [**V**].

2. Further applications. Besides the applications in [**A** and **Aa**], we mention two generalizations of the classical Borsuk-Ulam theorem: If S^n and

S^m have $\mathbf{Z}/2\mathbf{Z}$-actions and $m > n$, then there exists an equivariant map $f\colon S^m \to S^n$ if and only if $(S^n)^{\mathbf{Z}/2\mathbf{Z}} \neq \varnothing$.

THEOREM. *Suppose X and Y are connected G-spaces such that* $\dim Y < \infty$ *and $H_j(Y; \mathbf{F}_p) = 0$ for $j > n$ and $H_j(X; \mathbf{F}_p) = 0$ for $j \leq n$. Assume that G is p-elementary abelian. Then there exists an equivariant map $f\colon X \to Y$ if and only if $Y^G \neq \varnothing$.*

There is an infinite dimensional generalization also where V_G^r plays the role of isotropy subgroups

THEOREM. *Suppose $f\colon X \to Y$ is a G-map between connected G-spaces such that $\overline{H}_i(X; \mathbf{Z}/|G|\mathbf{Z}) = 0$ for $i \leq n$ and $H_j(Y; \mathbf{Z}/|G|\mathbf{Z}) = 0$ for $j > n$. Then for any $p|\,|G|$ and the corresponding kG-varieties, we have $V_G^r(Y) = V_G^r(k)$. (Similarly for kG-complexes.)*

There is also a localization theorem. For a p-elementary abelian group $\pi \cong (\mathbf{Z}/p\mathbf{Z})^n$, let $\gamma(\pi)$ be the product of polynomial generators in $H^*(\pi; k)$, and $\operatorname{rk}_p(G) = \max\{n\colon (\mathbf{Z}/p\mathbf{Z})^n \subseteq G\}$.

THEOREM. *Let $C_* \subseteq D_*$ be a pair of kG-complexes such that G-complexity of D_*/C_* is* $\operatorname{rk}_p(G) - s$. *Then there exists a subgroup $\pi \subset kG$, $\pi \cong (\mathbf{Z}/p\mathbf{Z})^s$, such that (localized hypercohomologies)*

$$\mathbf{H}^*(\pi, C^*)\left[\frac{1}{\gamma(\pi)}\right] \cong \mathbf{H}^*(\pi; D^*)\left[\frac{1}{\gamma(\pi)}\right].$$

There is a similar theorem for G-spaces (possibly infinite dimensional).

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