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SOME LOCAL-GLOBAL RESULTS IN FINITE TRANSFORMATION GROUPS

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ABSTRACT. The main theorem is a converse to the localization theorem in equivariant cohomology, using the Sullivan fixed point conjecture. In applications, we prove that certain fibrations over BG are fiber homotopy equivalent to the Borel constructions of finite dimensional (finitely dominated) G-spaces.

Introduction. Let G be a finite group, and let X be a G-space. The Borel equivariant cohomology of X is denoted by

$$H_G^*(X) \equiv H^*(E_G \times_G X; \mathbf{F}_p),$$

where $E_G \times_G X \xrightarrow{\pi} BG$ is the twisted product associated to the universal bundle $E_G \to BG$. For a p-elementary abelian group, i.e.

$$G \simeq (\mathbf{Z}/p\mathbf{Z})^n, \bigoplus_{k \geq 0} H^{2k}(BG; \mathbf{F}_p)_{\text{red}} \simeq \mathbf{F}_p[t_1, \ldots, t_n]$$

is a polynomial algebra denoted by H_G. Then $H_G^*(X)$ is a graded H_G-module and the localization $S^{-1}H_G^*(X)$ is obtained by inverting nonzero elements of $H^2(BG; \mathbf{F}_p)$. According to the Borel localization theorem [**B**], for a finite dimensional G-space X, the inclusion of the fixed points $j: X^G \to X$ induces an $S^{-1}H_G$- isomorphism $j^*: S^{-1}H_G^*(X) \xrightarrow{\simeq} S^{-1}H_G^*(X^G)$. Further refinements of localization in equivariant cohomology with deep applications are due to Quillen [**Q**] and W. Y. Hsiang [**Hs**], where the p-elementary abelian groups (and tori) play a key role in an algebro-geometric setting. In [**B, Hs, Q**] and related developments, the hypothesis $\dim X < \infty$ is indispensable since the localization theorem fails if $\dim X = \infty$. In [**Su**] D. Sullivan pointed to new directions. Let $\text{Map}(E_G, X)$ be the mapping space together with the conjugation G-action $f^g(x) = g^{-1}f(gx)$, so that the subspace of equivariant maps $\text{Map}_G(E_G, X)$ and the fixed point set $\text{Map}(E_G, X)^G$ coincide. In 1970 Sullivan conjectured that for $G = \mathbf{Z}/2\mathbf{Z}$, the inclusion $X^G \to \text{Map}_G(E_G, X)$ (via constant maps) induces a weak homotopy equivalence after 2-adic completion [**Su**]. In 1982, H. Miller's proof of the important special case $X = X^G$ marked a new era in transformation groups and homotopy theory [**M**]. As pointed out by H. Miller,

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the general form of the conjecture as stated in [**Su**] needs technical modification to be true. Recently, G. Carlsson [**C**], J. Lannes [**L**], and H. Miller [**Mm**] have announced independent proofs of modified versions of the Sullivan conjecture for $G = p$-group. This leads to a converse to the localization theorem and its corollaries below.

Notation and conventions. All spaces are paracompact. $p =$ prime integer, $\mathscr{P}_p =$ the set of p-subgroups of G;

$$\mathscr{C}_p = \{X\colon \dim X < \infty \text{ and } \dim H_*(X; \mathbf{F}_p) < \infty\};$$
$$\mathscr{C}_0 = \{X\colon X \text{ is weakly equivalent to some } Y \in \bigcap_{P| |G|} \mathscr{C}_p\};$$
$$\mathscr{C}_G = \{G\text{-space } X\colon \forall H \in \mathscr{P}_p \forall p| |G|, X^H \in \mathscr{C}_p \text{ and } X \in \mathscr{C}_p\};$$

$\mathscr{R}_p$ = a suitable p-completion functor such as [**BK**, **Mm**] and $\rho^X\colon X \to \mathscr{R}_p(X)$ the natural map; $\mathrm{Top}(\mathscr{R}_p) = \{X\colon H_*(\rho^X : \mathbf{F}_p)$ is an isomorphism$\}$= "p-good spaces" of [**BK**] if $\mathscr{R}_p$ is the Bousfield-Kan completion. E is a free contractible G-space, i.e. $E = E_G$.

1. A converse to the localization theorem. In problems involving the construction of G-actions (e.g. on manifolds or finite CW complexes), one often proceeds by homotopically constructing a fibration $Y \to BG$ with fiber homotopy equivalent to the given space X, so that the G-covering $\tilde{Y}$ has a free G-action and $\tilde{Y} \simeq X$. In practice, $\dim \tilde{Y} = \infty$ and one needs to know:

When does there exist a finite dimensional (finitely dominated) G-space K such that $\tilde{Y} \simeq_G E_G \times K$ (i.e. G-homotopy equivalent).

To construct a "finite dimensional model K" from the fibration $Y \to BG$, one encounters questions intimately related to the Sullivan conjecture. The valid statements of the Sullivan conjecture involve completion functors of Bousefield and Kan [**BK**] in [**C**, **L**] and the version of Dwyer-Miller-Neisendorfer in [**Mm**]. For $X \in \mathscr{C}_G$, let $F = X^G$. Let $\bar{\eta}\colon F \to \mathrm{Map}_G(E, X)$ and $\hat{\eta}$ = adjoint of the composition

$$E \times \mathscr{R}_p(F) \to \mathscr{R}_p(E) \times \mathscr{R}_p(F) \cong \mathscr{R}_p(E \times F) \xrightarrow{\mathscr{R}_p(\bar{\eta})} \mathscr{R}_p(X).$$

We need the following modified version of the Sullivan conjecture.

THEOREM (G. CARLSON, J. LANNES, H. MILLER). *Let $G = \mathbf{Z}_p^n$ and $X \in \mathscr{C}_G$ be connected. Then $\hat{\eta}\colon \mathscr{R}_p(F) \to \mathrm{Map}_G(E, \mathscr{R}_p(X))$ is a weak equivalence.*

The following provides a converse to the localization theorem as well as a characterization of finite dimensional G-spaces in terms of localized equivariant cohomology. There is also an analogous theorem for finitely dominated G-spaces.

THEOREM 1. *Let G be any finite group acting on a simply-connected G-space $X \in \mathscr{C}_0$. There exists a finite dimensional G-space $Y \in \mathscr{C}_G$ such that $E \times Y \simeq_G E \times X$ and $Y^H \in \mathrm{Top}(\mathscr{R}_p)$ $(\forall p \forall H \in \mathscr{P}_p)$ if and only if:*

(A1) *For each* $P \in \mathscr{P}_p$, *there exists* $F \in \mathrm{Top}(\mathscr{R}_p) \cap \mathscr{C}_p$ *and* $\eta\colon F \to \mathrm{Map}_p(E, X)$ *inducing an isomorphism*

$$\hat{\eta}_*\colon H_*(F; \mathbf{F}_p) \to H_*(\mathrm{Map}_p(E, X); \mathbf{F}_p).$$

(A2) *For each pair* $Q \subseteq P \in \mathscr{P}_p$ *with* $Q/P \equiv A \cong \mathbf{Z}_p^n$ *and inclusion* $j\colon \mathrm{Map}_p(E, \mathscr{R}_p(X)) \to \mathrm{Map}_Q(E, \mathscr{R}_p(X))$, $S^{-1}H_A^*(j)$ *is an isomorphism.*

IDEA OF PROOF. First, proceed for the case $G = \mathbf{Z}/p\mathbf{Z}$ by translating the Sullivan conjecture in terms of the localization theorem of Borel-Quillen-Hsiang for mapping spaces to obtain the necessary conditions. For the sufficiency, the space K is constructed directly starting with the fixed point set information contained in the space of sections of $E_G \times_G X \to BG$. The technical issues involve dealing with completions and localizations. For the general case, the singular set of the candidate K is constructed inductively, using algebraic computations and frequent applications of the projectivity criterion of [**A**, **Aa**] (in its modular form). Using the infinite dimensional G-space $\mathrm{Map}(E_G, X)$ as the target, we add free G-cells to the singular set to obtain finite dimensional approximations. The final stage reduces to proving the $\mathbf{Z}G$-projectivity of a certain module. According to [**A**, **Aa**], this can be detected by restriction to subgroups isomorphic to $\mathbf{Z}/p\mathbf{Z}$, and $(\mathbf{Z}/p\mathbf{Z})$-projectivity follows from the first case.

2. COROLLARY. *Let X be a G-space. Then the following are equivalent:*

(1) *There exists a finite dimensional (finitely dominated) G-space* $K \in \mathscr{C}_G$ *such that* $E \times X \simeq_G E \times K$.

(2) *For each representative p-Sylow subgroup* $P \subset G$, *there exists a finite dimensional (finitely dominated) P-space* $K(P) \in \mathscr{C}_p$ *such that* $E \times X$ *is P-homotopy equivalent to* $E \times K(P)$.

In some cases, condition (2) of the above can be further simplified by using only subgroups isomorphic to $\mathbf{Z}/p\mathbf{Z}$. Theorem 1 is useful in particular in conjunction with Lannes' T-functor [**L**] in circumstances where the fundamental groups of the related mapping spaces do not cause difficulties, e.g.:

3. COROLLARY. *Let X be a simply-connected G-space such that* $H_i(X) = 0$ *for* $i \neq 0, n$. *Suppose for each p-Sylow subgroup* $H \subseteq G$, $H_n(X)|_{\mathbf{Z}H} \cong P \oplus Q$, *where P is $\mathbf{Z}H$-projective, and Q is $\mathbf{Z}H$-indecomposable if* $|H| = p$ *and* $Q = 0$ *otherwise. Then there exists a G-space K,* $\dim K < \infty$, *such that* $E_G \times X \simeq_G E_G \times K$.

4. COROLLARY. *Let G be any finite group of order q. Suppose* $Y \xrightarrow{\eta} BG$ *is a fibration with fiber* $F \in \mathscr{C}_0$, *such that* $H_*(F; \mathbf{Z}/q) \cong H_*(S^n; \mathbf{Z}/q)$. *Then η is fiber homotopy equivalent to the Borel construction* $E_G \times_G K \to BG$ *where* $\dim K < \infty$ $(\pi_1(F) = 0)$.

The above corollaries use Theorem 1 to reduce the problem to p-elementary abelian groups. Then the results of Lannes [**L**] and other computations (e.g. of [**CLM**]) in conjunction with Lannes' T-functor allow one to verify the localization condition on the equivariant mapping spaces.

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