BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 19, Number 2, October 1988

CLASSIFICATION OF INVARIANT CONES IN LIE ALGEBRAS

JOACHIM HILGERT AND KARL H. HOFMANN

All *Lie algebras* in the following are finite dimensional real Lie algebras. A *cone* in a finite dimensional real vector space is a closed convex subset stable under the scalar multiplication by the set $\mathbf{R}^+$ of nonnegative real numbers; it is, therefore additively closed and may contain vector subspaces. A cone W in a Lie algebra $\mathfrak{g}$ is called *invariant* if

$$(1) \qquad e^{\operatorname{ad} x}(W) = W \quad \text{for all } x \in \mathfrak{g}.$$

We shall describe invariant cones in Lie algebras completely. For simple Lie algebras see [**KR82**, **Ol81**, **Pa84**, and **Vi80**].

Some observations are simple: *If W is an invariant cone in a Lie algebra $\mathfrak{g}$, then the edge $\mathfrak{e} = W \cap -W$ and the span $W - W$ are ideals.* Therefore, if one aims for a theory without restriction on the algebra $\mathfrak{g}$ it is no serious loss of generality to assume that W is *generating*, that is, satisfies $\mathfrak{g} = W - W$. This is tantamount to saying that W has inner points. Also, the homomorphic image $W/\mathfrak{e}$ is an invariant cone with zero edge in the algebra $\mathfrak{g}/\mathfrak{e}$. Therefore, nothing is lost if we assume that W is *pointed*, that is, has zero edge. Invariant pointed generating cones can for instance be found in $\mathrm{sl}(2, \mathbf{R})$, the oscillator algebra and compact Lie algebras with nontrivial center (see [**HH85b**, **c**, **HH86a**, or **HHL87**]).

A subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is said to be *compactly embedded* if the analytic group $\operatorname{Inn}_{\mathfrak{g}} \mathfrak{h}$ generated by the set $e^{\operatorname{ad} \mathfrak{h}}$ in $\operatorname{Aut} \mathfrak{g}$ has a compact closure. Even for a compactly embedded Cartan algebra $\mathfrak{h}$ of a solvable algebra $\mathfrak{g}$, the analytic group $\operatorname{Inn}_{\mathfrak{g}} \mathfrak{h}$ need not be closed in $\operatorname{Aut}_{\mathfrak{g}}$ [**HH86**]. An element $x \in \mathfrak{g}$ is called *compact* if $\mathbf{R} \cdot x$ is a compactly embedded subalgebra, and the set of all compact elements of $\mathfrak{g}$ will be denoted $\operatorname{comp} \mathfrak{g}$. It is true, although not entirely superficial that a *superalgebra is compactly embedded if and only if it is contained in* $\operatorname{comp} \mathfrak{g}$.

1. THEOREM (THE UNIQUENESS THEOREM [**HH86b**]). *Let W be an invariant pointed generating cone in a Lie algebra $\mathfrak{g}$. Then*

(i) $\operatorname{int} W \subseteq \operatorname{comp} \mathfrak{g}$.

(ii) *If H is any compactly embedded Cartan algebra, then*

(a) $H \cap \operatorname{int} W \neq \varnothing$, *and*

(b) $\operatorname{int} W = (\operatorname{Inn}_{\mathfrak{g}} \mathfrak{g}) \operatorname{int}_{\mathfrak{h}}(\mathfrak{h} \cap W)$.

In particular, compactly embedded Cartan algebras exist, and if $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are compactly embedded Cartan algebras and W_1 and W_2 are invariant pointed generating cones of $\mathfrak{g}$ such that $\mathfrak{h} \cap W_1 = \mathfrak{h} \cap W_2$, then $W_1 = W_2$. □

Received by the editors December 16, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 22E60, 22E15.

This result shows that we know W if we know $\mathfrak{h} \cap W$ for any compactly embedded Cartan algebra $\mathfrak{h}$.

We consider a compactly embedded Cartan algebra $\mathfrak{h}$ and denote by Γ the torus $\overline{\operatorname{Inn}_{\mathfrak{g}} \mathfrak{h}}$. Then we obtain the linear projection operator $P: \mathfrak{g} \to \mathfrak{g}$ by $P(x) = \int_\Gamma g(x)\,dg$ with normalized Haar measure on Γ. Then $\mathfrak{h} = P(\mathfrak{g})$ and $\mathfrak{g}$ decomposes into a direct sum of $\mathfrak{h}$-modules $\mathfrak{h} \oplus \mathfrak{h}^+$ with $\mathfrak{h}^+ \stackrel{\text{def}}{=} \ker P$. *For an invariant cone W and any compactly embedded Cartan algebra $\mathfrak{h}$ the meet $\mathfrak{h} \cap W$ and the projection $P(W)$ are related by*

$$P(W) = \mathfrak{h} \cap W. \tag{2}$$

If C is a pointed cone in a compactly embedded Cartan algebra $\mathfrak{h}$ we define a cone in $\mathfrak{g}$ by

$$\tilde{C} = \bigcap_{g \in \operatorname{Inn}_{\mathfrak{g}} \mathfrak{g}} gP^{-1}(C). \tag{3}$$

Then $\tilde{C} = \{x \in \mathfrak{g} | P((\operatorname{Inn}_{\mathfrak{g}} \mathfrak{g})x) \subseteq C\}$ and $\tilde{C}$ is an invariant cone in $\mathfrak{g}$. Its edge is the largest ideal of $\mathfrak{g}$ contained in $\mathfrak{h}^+$. It is not a seriously restrictive assumption that H^+ should not contain nonzero ideals. Under these circumstances, unfortunately, $\tilde{C}$ may be zero. However, the following theorem uses the device $\tilde{C}$ to reconstruct W from $\mathfrak{h} \cap W$:

2. THEOREM (THE RECONSTRUCTIONS THEOREM **[HH86b]**). *Suppose that $\mathfrak{h}$ is a compactly embedded Cartan algebra $\mathfrak{h}$ such that $\mathfrak{h}^+$ contains no nonzero ideal of $\mathfrak{g}$. If C is a pointed generating cone in $\mathfrak{h}$ then the following statements are equivalent:*

(A) *There exists an invariant pointed cone W in L such that $C = \mathfrak{h} \cap W$.*

(B) $C = \mathfrak{h} \cap \tilde{C}$.

(C) *Each conjugacy class of an element $c \in C$ projects into C under P.*

Moreover, if these conditions are satisfied, then $W = \tilde{C}$. □

The problem is now to determine which cones C satisfy condition (C) of Theorem 1 and in which Lie algebras they can occur.

3. PROPOSITION **[HH86]**. *Every compactly embedded Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is contained in a unique maximal compactly embedded subalgebra $\mathfrak{k}(\mathfrak{h})$. A subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is maximal compactly embedded if and only if* $\operatorname{INN}_{\mathfrak{g}} \mathfrak{k} \stackrel{\text{def}}{=} \overline{\operatorname{Inn}_{\mathfrak{g}} \mathfrak{k}}$ *is a maximal compact subgroup of* $\operatorname{INN} \mathfrak{g}$. □

Under the circumstances of Proposition 3, the normalizer $N(\mathfrak{h})$ of the maximal torus $\Gamma = \operatorname{INN}_{\mathfrak{g}} \mathfrak{h}$ in $\operatorname{INN} \mathfrak{g}$ is contained in the compact subgroup $K(\mathfrak{h}) = \operatorname{INN}_{\mathfrak{g}} \mathfrak{k}(\mathfrak{h})$. Thus $N(\mathfrak{h})/\Gamma$ is a finite group, called the *Weyl group $\mathscr{W}$ of the pair* $(\mathfrak{g}, \mathfrak{h})$. The space $\mathfrak{h}^+$ is a Γ-module for the torus Γ and thus decomposes into isotypic components. The search for an appropriate natural indexing for such an isotypic component $\mathfrak{v}$ leads to a real linear form $\omega: \mathbf{h} \to \mathbf{R}$ and a complex structure $I_\omega: \mathfrak{h}^+ \to \mathfrak{h}^+$ (that is, a vector space automorphism with $I_\omega^2 = -1$) such that the $\mathfrak{h}$-module structure of $\mathfrak{v}$ is given by

$$[h, x] = \omega(h) \cdot I_\omega(x).$$

We define

$$\mathfrak{g}^\omega = \{x \in \mathfrak{g} | (\exists I_\omega) I_\omega^2 = -1 \text{ and } (\forall h \in \mathfrak{h})\ [h, x] = \omega(h) \cdot Ix\}.$$

We let Ω denote the set of all ω for which $\mathfrak{g}^\omega \neq \{0\}$ and call these linear forms on $\mathbf{h}$ the real roots of the pair $(\mathfrak{g}, \mathfrak{h})$. We note $\mathfrak{g}^0 = \mathfrak{h}$. Any choice of a closed half space E in the dual $\hat{\mathfrak{h}}$ of $\mathfrak{h}$ whose boundary hyperplane meets the finite set Ω only in 0 allows us to represent Ω as a union $\Omega = \Omega^+ \cup -\Omega^+$ with $\Omega^+ = \Omega \cap E$. We shall call Ω^+ *a selection of positive roots* and find the *real roots decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^+, \qquad \mathfrak{h}^+ = \sum_{0 \neq \omega \in \Omega^+} \mathfrak{g}^\omega, \tag{4}$$

of $\mathfrak{g}$ *with respect to* $\mathfrak{h}$. The family of complex structures I_ω on $\mathfrak{g}^\omega$ then, once a selection of positive roots has been made, gives a complex structure I on $\mathfrak{h}^+$ with which the bracketing of elements from $\mathfrak{h}$ with those from any $\mathfrak{g}^\omega$ is described by

$$[h, x] = \omega(h) \cdot Ix \quad \text{for all } x \in \mathfrak{g}^\omega. \tag{5}$$

At a later point it is important to have available certain special selections of positive roots.

The complex structure I on $\mathfrak{g}^+$ allows us to define a quadratic function

$$Q\colon \mathfrak{h}^+ \to \mathfrak{h}, \quad Q(x) = P([Ix, x]). \tag{6}$$

For $0 \neq \omega \in \Omega^+$ and $x \in \mathfrak{g}^\omega$ we have

$$Q(x) = [Ix, x] = -[x, Ix]. \tag{7}$$

Keep in mind that Q depends on the selection of a set of positive roots via I. Changing such a selection may change $Q(x)$ by a sign.

4. PROPOSITION [**HH86b**, **HHL87**]. *If* $\mathfrak{g}$ *accommodates an invariant pointed generating cone and* $\mathfrak{h}$ *is a compactly embedded Cartan algebra, then* $Q(x) = 0$ *and* $x \in \mathfrak{g}^\omega$ *imply* $x = 0$. □

This motivates the following definition.

5. DEFINITION. A Lie algebra $\mathfrak{g}$ is said *to have cone potential* if it has a compact embedded Cartan algebra $\mathfrak{h}$ and $0 \neq x \in \mathfrak{g}^\omega$ for any positive real rootω implies $Q(x) \neq 0$.

The structure of Lie algebras with cone potential is special:

6. THEOREM. *Let $\mathfrak{g}$ be a Lie algebra with cone potential, $\mathfrak{h}$ a compactly embedded Cartan algebra, $\mathfrak{r}$ its radical, $\mathfrak{n}$ is nilradical, $\mathfrak{z}$ its center. Let Ω^+ be any selection of positive real roots with respect to $\mathfrak{h}$. For any $\mathfrak{h}$-submodule $\mathbf{v}$ of $\mathfrak{g}$ we write $\mathbf{v}^\omega = \mathbf{v} \cap \mathfrak{g}^\omega$. Then the following conclusions hold:*

(i) *$\mathfrak{z}$ is the center of $\mathfrak{n}$ and $\mathfrak{n}/\mathfrak{z}$ is abelian.*

(ii)

$$[\mathfrak{n}^\omega, \mathfrak{n}^{\omega'}] \begin{cases} \neq \{0\}, & \text{if } \omega = \omega'; \\ = \{0\}, & \text{if } \omega \neq \omega'. \end{cases}$$

(iii) $\mathfrak{r}^\omega = \mathfrak{n}^\omega$ *for* $0 \neq \omega \in \Omega^+$.

(iv) *There is a Levi complement $\mathfrak{s}$ such that*

$$\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{r}) \oplus (\mathfrak{h} \cap \mathfrak{s}),$$

and $\mathfrak{h} \cap \mathfrak{s}$ is a compactly embedded Cartan algebra of $\mathfrak{s}$.

(v) $[\mathfrak{h}, \mathfrak{s}] \subseteq \mathfrak{s}$ *and* $\mathfrak{h} + \mathfrak{s} = (\mathfrak{h} \cap \mathfrak{r}) \oplus \mathfrak{s}$ *is a reductive subalgebra.*

(vi) $\mathfrak{g}^\omega = \mathfrak{r}^\omega \oplus \mathfrak{s}^\omega$ *for* $\omega \in \Omega^+$. □

However, Lie algebras supporting invariant cones are even more special.

7. PROPOSITION **[HH86b]**. *Let W be an invariant pointed generating cone in $\mathfrak{g}$ and let $\mathfrak{h}$ be a compactly embedded Cartan algebra. Then the center $\mathfrak{c}$ of the unique maximal compactly embedded subalgebra $\mathfrak{k}(\mathfrak{h})$ containing $\mathfrak{h}$ contains inner points of* comp $\mathfrak{g}$. *Moreover, the centralizer of $\mathfrak{c}$ in $\mathfrak{g}$ is $\mathfrak{k}(\mathfrak{h})$.* □

Such phenomena occur in the context of hermitean symmetric spaces inside semisimple Lie algebras. This motivates the following notation:

8. DEFINITION. A Lie algebra $\mathfrak{g}$ is called *quasihermitean* if it contains a compactly embedded Cartan algebra $\mathfrak{h}$ such that the center $\mathfrak{c}$ of $\mathfrak{k}(\mathfrak{h})$ satisfies

$$\mathfrak{c} \cap \operatorname{int}(\operatorname{comp} \mathfrak{g}) \neq \varnothing. \tag{8}$$

Recalling that $\mathfrak{z}(x) = \ker \operatorname{ad} x$ is the centralizer of x in $\mathfrak{g}$, one shows that

$$\mathfrak{c} \cap \operatorname{int}(\operatorname{comp} \mathfrak{g}) = \{x \in \mathfrak{g} | \mathfrak{z}(x) = \mathfrak{k}(\mathfrak{h})\}. \tag{9}$$

9. DEFINITION. Let Ω be the set of real roots of a quaishermitean Lie algebra $\mathfrak{g}$ with respect to a compactly embedded Cartan algebra $\mathfrak{h}$. Then $\omega \in \Omega$ is said to be a *compact root* if $\mathfrak{g}^\omega \subseteq \mathfrak{k}(\mathfrak{h})$. All other roots are *noncompact*. The set of compact roots is denoted Ω_k, the complement is Ω_p. For any selection of positive roots Ω^+ we set $\Omega_k^+ = \Omega^+ \cap \Omega_k$ and $\Omega_p^+ = \Omega^+ \cap \Omega_p$. Finally, we set

$$\mathfrak{p}(\mathfrak{h}) = \bigoplus_{\omega \in \Omega_p^+} \mathfrak{g}^\omega. \tag{10}$$

For any choice of an element $c \in \mathfrak{c} \cap \operatorname{int}(\operatorname{comp} \mathfrak{g})$ there is a selection Ω^+ of positive roots such that $\omega(c) > 0$ for all noncompact roots ω.

10. THEOREM. *Let $\mathfrak{g}$ denote a quasihermitean Lie algebra and fix a compactly embedded Cartan algebra $\mathfrak{h}$. Let $\mathfrak{r}$ denote the radical. Then the following* *conclusions hold:*

(i) $\mathfrak{k}(\mathfrak{h}) = \mathfrak{h} \oplus \bigoplus_{0\neq\omega\in\Omega_k^+} \mathfrak{g}^\omega$.

(ii) $\mathfrak{g} = \mathfrak{k}(\mathfrak{h}) \oplus \mathfrak{p}(\mathfrak{h})$ *and* $[\mathfrak{k}(\mathfrak{h}), \mathfrak{p}(\mathfrak{h})] \subseteq \mathfrak{p}(\mathfrak{h})$.

(iii) *The unique largest ideal of* $\mathfrak{g}$ *contained in* $\mathfrak{p}(\mathfrak{h})$ *contains all ideals* $\mathfrak{i}$ *with* $\mathfrak{h} \cap \mathfrak{i} = \{0\}$.

(iv) $\mathfrak{r} \subseteq \mathfrak{h} \oplus \mathfrak{p}(\mathfrak{h})$.

(v) *Let* $c \in \mathfrak{c} \cap \operatorname{int}(\operatorname{comp}\mathfrak{g})$ *and let* Ω^+ *be a selection of positive roots such that* $\omega(c) > 0$ *for all* $\omega \in \Omega_p^+$. *Then, with respect to the complex structure* $I|\mathfrak{p}(\mathfrak{h})$, *the vector space* $\mathfrak{p}(\mathfrak{h})$ *is a complex* $k(\mathfrak{h})$*-module, i.e.,* $[k, Ip] = I[k,p]$. □

It is not hard to record some necessary conditions for a pointed generating cone C in $\mathfrak{h}$ to be of the form $W \cap \mathfrak{h}$. The first is immediate from the definitions

(WEYL) $$\mathscr{W} C = C.$$

A detailed analysis of the orbits of an element $h \in \mathfrak{h}$ under a one-parameter group of inner automorphisms $e^{\mathbf{R}\cdot \operatorname{ad} x}$ for a root vector $x \in \mathfrak{g}^\omega$ reveals another necessary condition.

For each nonzero real root $\omega \in \Omega$ we define a function $Q_\omega \colon \mathfrak{h} \times \mathfrak{g}^\omega \to \mathfrak{h}$ by $Q_\omega(h, x) = \omega(h)\cdot Q(x) = \omega(h)\cdot[Ix, x] = \omega(h)\cdot[I_\omega x, x]$. While I and Q depend on a selection of positive roots, the functions Q_ω do not. If $C = \mathfrak{h} \cap W$ for an invariant pointed generating cone W, then we find $Q_\omega(C \times \mathfrak{g}^\omega) \in C$ for all $\omega \in \Omega_p$.

This condition is equivalent to

(ROOT) $$(\operatorname{ad} x)^2 C \subseteq C \quad \text{for all } x \in L^\omega,\ \omega \in \Omega_p.$$

The main result is that the two conditions (WEYL) and (ROOT) are also sufficient for C to be of the form $\mathfrak{h} \cap W$.

11. THEOREM (THE MAIN CHARACTERISATION THEOREM). *Let* $\mathfrak{g}$ *denote a quasihermitean Lie algebra with cone potential, and let* $\mathfrak{h}$ *be a compactly embedded Cartan algebra. Let* C *be a pointed generating cone in the vector space* $\mathfrak{h}$. *Then there exists a unique invariant pointed generating cone* W *in* $\mathfrak{g}$ *if and only if conditions* (WEYL) *and* (ROOT) *are satisfied.* □

REFERENCES

[**HH85a**] J. Hilgert and K. H. Hofmann, *Lorentzian cones in real Lie algebras*, Monatsh. Math. **100** (1985), 183–210.

[**HH85b**] ——, *Old and new on* Sl(2), Manuscripta Math. **54** (1985), 17–52.

[**HH85c**] ——, *Lie semialgebras are real phenomena*, Math. Ann. **270** (1985), 97–103.

[**HH86a**] ——, *On the automorphism group of cones and wedges*, Geom. Dedicata **21** (1986), 205–217.

[**HH86b**] ——, *Compactly embedded Cartan algebras and invariant cones in Lie algebras*, THD preprint (1986), Adv. in Math. (to appear).

[**HHL87**] J. Hilgert, K. H. Hofmann and J. D. Lawson, *Lie groups, convex cones, and semigroups*, Oxford Univ. Press (to appear).

[**KR82**] S. Kumaresan and A. Ranjan, *On invariant convex cones in simple Lie algebras*, Proc. Indian Acad. Sci. Math. **91** (1982), 167–182.

[**Ol81**] G. I. Ol'shanskiĭ, *Invariant cones in Lie algebras, Lie semigroups and the holomorphic discrete series*, Funct. Anal. Appl. **15** (1981), 275–285.

[**Pa84**] S. Paneitz, *Determination of invariant convex cones in simple Lie algebras*, Ark. Mat. **21** (1984), 217–228.

[**Vi80**] E. B. Vinberg, *Invariant cones and orderings in Lie groups*, Funct. Anal. Appl. **14** (1980), 1–13.

FACHBEREICH MATHEMATIK, TECHNISCHE HOCHSCHULE DARMSTADT, SCHLOSSGARTENSTR. 7, D-6100 DARMSTADT, FEDERAL REPUBLIC OF GERMANY

Electronic address: XMATDA4L @DDATHD21.bitnet