

systems by relating them to the theory of Lie groups, Lie algebras and symmetric spaces. A Lie group  $G$  acts on the dual  $g^*$  of its Lie algebra and every orbit in  $g^*$  has a canonical symplectic structure. It is conjectured in general and proved in many special cases that there are smooth functions  $f_1, \dots, f_k$  on  $g^*$  which restrict to a full commutative set of functions on generic orbits in  $g^*$ . Thus the Hamiltonian systems on the orbits defined by any of the functions  $f_i$  can be solved completely. In the last chapter of his book Fomenko shows how several differential systems arising from classical mechanics (such as the motion of a solid body in an ideal fluid) can be solved in this way by virtue of their equivalence to Hamiltonian systems on orbits in duals of Lie algebras.

Unfortunately this book suffers from having been poorly translated from Russian. Indeed a basic knowledge of the Russian language makes reading easier. In most cases the poor translation is merely a source of amusement or at worst irritation. However some errors are more serious, such as when the phrase “*must not be*” is used where “*is not necessarily*” is meant. These errors are likely to cause unnecessary confusion to those at whom the book is aimed, that is, graduate students and nonspecialists learning the subject for the first time.

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*Pseudodifferential operators and spectral theory*, by M. A. Shubin. Translated by Stig I. Andersson. Springer-Verlag, Berlin, Heidelberg, New York, 1987, x + 278 pp., \$55.00. ISBN 3-540-13621-5

The publishers are to be commended for making this text by M. A. Shubin accessible to the mathematicians who do not read Russian. It contains a fairly short, yet highly readable account of pseudodifferential, and Fourier integral, operator theory, with extensive applications to the spectral theory of linear elliptic equations. Let me say right away that any mathematician tempted to give a first course in the subject of  $\Psi$ DO and/or FIO should give serious consideration to Chapter I, plus Appendix I, of Shubin's book as a possible text. They present most of what is needed for a basic understanding of the theory in a style that is simple yet precise. They are independent of the other chapters, to which the reader might want to go for significant applications and extensions.

There is nothing novel in the application of pseudodifferential operators to elliptic problems. Elliptic problems are one of the main sources from which the

theory sprang in the early sixties (in the fifties a primitive version of Fourier integral operators was applied to hyperbolic problems by P. D. Lax). The usefulness of  $\Psi$ DO in the proof of the Atiyah-Singer index theorem gave them much of their initial impetus. The flexibility they afforded was exploited, early on, for instance by Seeley (in [See]) to define the nonintegral (and complex) powers of elliptic operators, a topic that makes up the content of Chapter II of Shubin's book.

The symbolic calculus of pseudodifferential operators enables one to replace the direct analysis of a differential operator  $P(x, D)$  by manipulations of its *symbol*  $P_m(x, \xi) + P_{m-1}(x, \xi) + \cdots$  (the individual terms  $P_j(x, \xi)$  are homogeneous of degree  $j$  with respect to  $\xi$ ;  $m$  is the *order* of the differential operator). When the operator is elliptic and self-adjoint its *principal symbol*  $P_m(x, \xi)$  is real and different from zero for any  $\xi \neq 0$ . It is then natural to look at the symbol

$$P_m(x, \xi)^c \left\{ 1 + \sum_{j \geq 1} P_{m-j}(x, \xi)/P_m(x, \xi) \right\}^c$$

for a complex number  $c$ . For large  $|\xi|$  the sum over  $j \geq 1$  is small in comparison to 1 and the binomial expansion yields an infinite series of terms that are homogeneous with respect to  $\xi$  and whose homogeneity degrees decrease to  $-\infty$ . This is exactly what is called a *classical symbol* in pseudodifferential operator theory. It is the symbol of a pseudodifferential operator which can be equated to the power  $P(x, D)^c$  (defined, say, by the spectral decomposition of unbounded selfadjoint operators in the space  $L^2$  of the Riemannian manifold in which  $P$  operates). The standard Fourier integral representation by means of the symbol only approximates  $P(x, D)^c$  but the error is a smoothing operator, i.e., transforms any distribution into a  $\mathcal{C}^\infty$  function. On any compact manifold an operator with the latter property is compact, and much more than compact. One can also replace  $P(x, D)$  by  $P(x, D) - \lambda I$  and take  $c = -1$ ; with these choices the preceding construction will yield a good approximation of the resolvent of  $P(x, D)$ , thus opening the door to its spectral analysis.

From the standpoint adopted by Shubin another turning point came in 1968 when Hörmander showed (in [Hö1]) how to use a certain kind of linear operators, which he called Fourier integral operators, to derive an asymptotic estimate of the number  $N(\lambda)$  of eigenvalues not exceeding a number  $\lambda > 0$ , of a selfadjoint elliptic differential operator (of order  $m \geq 2$ ) on a compact manifold (without boundary). Later the approach was pushed further by J. Chazarain, V. Guillemin, A. Weinstein, et al. to tackle problems in Riemannian geometry, among them the Poisson formula relating the spectral function of the Laplace-Beltrami operator to the distribution of the lengths of the closed geodesics on a compact manifold.

The approach in [Hö1] was to combine the Tauberian theorems that had been used in this connection since earlier times, with the approximation of the spectral function made possible by the Fourier integral operators. There, of course, lay the startling novelty of the method. Chapter II of the book under review gathers a wealth of preliminary information on the resolvent

and the spectral function of the elliptic operator. The full implementation of Hörmander's method comes in Chapter III.

Let  $P = P(x, D)$  be, as before, an elliptic, selfadjoint differential operator. One can then represent the unitary group  $e^{itP}$  ( $t \in \mathbf{R}$ ) as a Fourier integral operator (again the abstract operator  $e^{itP}$  may be defined by spectral decomposition in the space  $L^2$  of the base manifold). When  $P$  is *positive* an alternative approach is to represent the semigroup  $e^{-tP}$  ( $t \geq 0$ ) by a FIO with *complex phase*. When the base manifold is compact the spectrum of  $P$  is discrete. The spectral measure  $dE^P(\lambda)$  of the selfadjoint operator  $P$  may be regarded as the Fourier transform of the function  $t \rightarrow e^{itP}$ , since

$$e^{itP} = \int_{-\infty}^{+\infty} e^{it\lambda} dE^P(\lambda).$$

Tauberian theorems can then be used to derive information about the behaviour of  $dE^P(\lambda)$  for  $|\lambda| \rightarrow +\infty$  from that of  $e^{itP}$  as  $t \rightarrow 0$ . Especially amenable to this approach are the *traces* of the operators, and it is important to ascertain that the operators are trace-class.

Chapter IV is concerned with elliptic equations on  $\mathbf{R}^n$ . Needless to say, elliptic equations on *noncompact* manifolds present more hurdles than those on compact manifolds without a boundary. On  $\mathbf{R}^n$  the role of boundary conditions is played, to some extent, by conditions on growth of the solutions at infinity. Analyzing such problems requires a more precise choice of pseudodifferential operators. Here, as in other questions, the Weyl calculus (see [Hö2]), as well as the anti-Wick symbol introduced by Berezin in [Ber], are more convenient than the standard calculus. A clear account of these new symbolic calculi is given in Chapter IV (in conjunction with Appendices 2 and 3), as well as a description of the classes of  $\Psi$ DO appropriate to the analysis in  $\mathbf{R}^n$ . It is of the essence, on noncompact manifolds, to make sure that the operators one makes use of belong to the trace-class (§27 and Appendix 3). Once fine-tuned, these tools enable one to establish, under suitable hypotheses, the discreteness of the spectrum (§26) and make possible the construction of approximate spectral projections (§28) and the asymptotic estimate of the number of eigenvalues (§30).

Related to the main subject of Shubin's book but not touched upon by it are elliptic *boundary value* problems (the recent monograph [Gr] covers extensively elliptic *pseudodifferential* boundary value problems). Nor does the book touch upon nonelliptic problems, such as diffraction theory, revolutionized in the late seventies by the new tools of microlocal analysis. Mathematicians whose interest lies more in those directions ought to look at other books, for instance [Tay] or [Hö3].

To repeat, the book by Shubin is a nice introduction to microlocal analysis, with emphasis on its application to spectral theory for linear elliptic equations on compact manifolds without a boundary. It is well written, in simple and direct language. In Chapter I each section is interspersed with simple exercises; some of the sections are followed by subsections describing problems for the reader. Some of these are difficult; the answers to the more difficult problems can be found in technical articles, to which precise reference is given. In the later chapters the number of exercises and problems decreases, as is to be expected in a text at a fairly advanced graduate level.

At the close of the book the reader will find historical notes that, in the main, seem to be accurate, and a comprehensive bibliography, as well as a Subject Index and an Index of Notation. Stig I. Andersson has done a superb job of translating the Russian text into English.

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*A treatise on generating functions*, by H. M. Srivastava and H. L. Manocha. Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, and Toronto, 1984, 569 pp., \$89.95. ISBN 0-470-20010-3

In the beginning was Laplace, who considered, among other things,

... ce que je nomme *fonctions génératrices*: c'est un nouveau genre de calcul que l'on peut nommer *calcul des fonctions génératrices*, et qui m'a paru mériter d'être cultivé par les géomètres. [2]

In other words, the idea that the two-variable function  $F$  is a *generating function* for the set of one-variable functions  $f_0, f_1, f_2, \dots$  by the way of the *generating function relation*

$$(1) \quad F(x, t) = \sum_{n=0}^{\infty} f_n(x)t^n,$$

was born just over two centuries ago. Legendre [3] gave, incidentally, one of the first examples: the generating function relation

$$(2) \quad (1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$$

for the Legendre polynomials  $P_0, P_1, P_2, \dots$ , known to most of us.