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ZERO-CYCLES, SPLITTING OF PROJECTIVE MODULES AND NUMBER OF GENERATORS OF A MODULE

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Let A denote (for this entire note) a reduced affine ring of dimension n over an algebraically closed field k. Theorem 1 below is about the existence of certain projective modules of rank n. We use this theorem to extend results in [**MKM**] to all dimensions.

I am thankful to Gennady Lyubeznik for providing motivating force and help in preparing this announcement.

Let $F^nK_0(A)$ denote the subgroup of $K_0(A)$ generated by the images of residue fields of all regular maximal ideals of height n. For a projective A-module P, we denote by (P) its image in $K_0(A)$. Let $I \subset A$ be an ideal such that I/I^2 is generated by n elements. We say that an ideal J is *residual* to I if

(i) $I + J = A$;

(ii) J is a local complete intersection of height n;

(iii) IJ is generated by n elements.

By general position arguments, there always exist such J which are products of regular maximal ideals of height n.

THEOREM 1. *Let I be an ideal in A such that I/I^2 is generated by n elements. Then there exists a projective A-module P of rank n and a surjection $P \twoheadrightarrow I$ such that $z = (P) - (A^n) \in F^nK_0(A)$. In fact, for any ideal J residual to I, there exists a P such that $(n-1)!\,z = (A/J)$.*

SKETCH OF PROOF. Using a souped-up version of [**MK**, Theorem 2] we may successively replace I by an ideal J residual to I. Thus we may assume I is a product of regular maximal ideals M_i. Now using the divisibility of the Picard group of a smooth complete intersection curve $\operatorname{Spec} A/(x_1, \ldots, x_{n-1})$, through the M_i, we find a product of regular maximal ideals N such that $L = N^{(n-1)!} + \sum_{i=1}^{n-1} Ax_i$ is residual to I. We replace I by L and finish the proof with the following lemma.

LEMMA. *Let A be a noetherian ring and I an ideal which is a local complete intersection of height r. Suppose I/I^2 is A/I-free and let $x_i \in I$, $1 \le i \le r$, generate I/I^2. Let $J = I^{(r-1)!} + (x_1, \ldots, x_{r-1})$. Then there exists a projective A-module P of rank r and a surjection $P \twoheadrightarrow J$ such that $(P) - (A^r)$ in $K_0(A)$.*

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For the proof of the lemma, we observe that it suffices to do the case when A is the Kumar-Nori quadric:

$$A = \mathbf{Z}[x_1, \ldots, x_r, y_1, \ldots, y_r, z]\colon\ z(z-1) + \sum x_i y_i = 0$$

and

$$I = (x_1, \ldots, x_r, z).$$

We finish the proof using [**Bo**] and a simple Chern class computation using Riemann-Roch. Using Theorem 1 and the fact that when A is regular, $F^n K_0(A)$ is a torsion-free divisible group [**Sr**], we get

THEOREM 2. *With the notation as in Theorem* 1, *further assume that A is regular. Then P is unique (depends only on I) up to isomorphism and*

$$(P) - (A^n) = -\frac{1}{(n-1)!} S(I),$$

where $S(I)$ is the image in $F^n K_0(A)$ of the zero-dimensional Segre class of $V(I)$ in Spec A.

For a definition of Segre class see [**Fu**, p. 13].

THEOREM 3. *Let $I \subset A$ be a local complete intersection of height n. Suppose $F^n K_0(A)$ has no $(n-1)!$-torsion. Then $(A/I) = 0$ in $K_0(A)$ if and only if I is a complete intersection.*

PROOF. Easy consequence of Theorem 1 and the Suslin cancellation theorem [**Su**].

REMARK. $F^n K_0(A)$ is known to be torsion-free in the following cases: char $k = 0$ [**L**], A is regular and $n = 2$ or $n \geq 3$, and A is regular in codimension one [**Sr**]. Also the torsion subgroup of $F^n K_0(A)$ is p-primary if char $k = p > 0$ [**L**].

THEOREM 4. *Suppose $F^n K_0(A)$ has no $(n-1)!$-torsion. Let P be a projective A-module of rank n. Then P has a free direct summand of rank one if and only if the nth Chern class $C_n(P)$ of P is zero.*

Recall that for a generic section $P \twoheadrightarrow I$, I a local complete intersection of height n, $C_n(P) = (A/I) \in F^n K_0(A)$.

PROOF. Take a generic section $P \twoheadrightarrow I$. Apply Theorem 3 and an argument in [**MK**].

Gennady Lyubeznik has pointed out the following application of Theorem 1.

THEOREM 5. *Suppose* char $k = p > 0$ *and P is a projective A-module of rank n such that $C_n(P)$ is torsion. Then for some r, the rth Frobenius iterate $P^{(r)}$ has a free direct summand of rank one.*

Eisenbud-Evans estimates. For a finite A-module M, let $\mu(M)$ (resp. $\nu(M)$) denote the maximum of $\mu_p(M) + \dim A/p$, where p runs through all $p \in \operatorname{supp} M$ (resp. $p \in \operatorname{supp} M$, $\dim A/p < \dim A$). It is well known that M is generated by $\mu(M)$ elements [**Fo**]. As an application of Theorem 1 we have

THEOREM 6. *Let M be a finite A-module. Then there exists a projective A-module P of rank $\nu(M)$ and a surjection $P \twoheadrightarrow M$ such that $(P)-(A^{\nu(M)}) \in F^n K_0(A)$.*

COROLLARY. *$F^n K_0(A) = 0$ if and only if M is generated by $\nu(M)$ elements, for all finite A-modules M.*

THEOREM 7. *Let A be regular and M a finite A-module. Then either $\mu(M) = \nu(M)$ or there exists a P as in Theorem 6, unique up to isomorphism. Further*

$$(P) - (A^{\nu(M)}) = -\frac{1}{(n-1)!} S_0(M),$$

where M is the zero-dimensional Segre class of M (see [**Fu**]*).*

COROLLARY. *Suppose A is regular. Then M is generated by $\nu(M)$ elements if and only if $\mu(M) = \nu(M)$ or $S_0(M) = 0$.*

The proofs of all these results rely heavily upon methods used in [**MKM** and **MK**]. We also remark that the lemma of this note can be used to show that $F^n K_0(A)$ is torsion-free if $n \geq 3$ and A is regular in codimension one. Complete proofs will appear elsewhere.

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