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EXAMPLES OF COMPLETE MANIFOLDS OF POSITIVE RICCI CURVATURE WITH NILPOTENT ISOMETRY GROUPS

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It is well known [**4**] that the isometry group of a complete riemannian manifold M with strictly positive sectional curvature is always compact. This is no longer true in general when M has Ricci curvature $\mathrm{Ric} > 0$. The first example was given in [**7**] for $\dim M = 4$. In this note we shall prove

THEOREM. *Let L be an n-dimensional simply connected nilpotent Lie group. Then for all sufficiently large p, the product manifold $M^{p+n} = \mathbf{R}^p \times L$ admits complete riemannian metrics with strictly positive Ricci curvature such that the isometry group of M contains L.*

Using a theorem of Malcev [**8**], we have as an immediate consequence:

COROLLARY. *Every finitely generated torsion-free nilpotent group can be realized as the fundamental group of a complete riemannian manifold with strictly positive Ricci curvature.*

On the other hand, every finitely generated subgroup of the fundamental group of any complete manifold with $\mathrm{Ric} \geq 0$ ($K \geq 0$) is nilpotent (abelian) up to finite index [**6**, **5**, **4**].

PROOF OF THE THEOREM. Our construction is inspired by [**2**]. We first apply an observation in [**3**, pp. 126–127] to obtain a family of almost flat metrics g_r on L, $0 \leq r < \infty$.

Choose a triangular basis $\{X_1, \ldots, X_n\}$ for the Lie algebra l of L, i.e., $[X, X_i] \in l_{i-1}$ whenever $X \in l$, and l_{i-1} is spanned by $X_1, \ldots, X_{i-1}$. For $X = \sum_{i=1}^n a_i X_i$ set $\|X\|^2 = \sum_{i=1}^n h_i^2(r) a_i^2$, where $h_i(r) = (1+r^2)^{-\alpha_i}$, and $\alpha_n = \alpha > 0$, $2\alpha_i - 4\alpha_{i+1} = 1$, $1 \leq i \leq n-1$. The above norm gives rise to a corresponding almost flat left invariant metric g_r. Then

$$|\operatorname{Ric}_L(X_i)| \leq c(1+r^2)^{-1}, \tag{1}$$

where c is a constant depending on n and the structure constants.

Now we define a warped product metric g on M by

$$g = dr^2 + f^2(r)\, ds^2 + g_r,$$

where ds^2 is the canonical euclidean metric on the sphere $S^{p-1} \subset \mathbf{R}^p$, $f(r) = r(1+r^2)^{-1/4}$. g is a complete metric on M, since $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f(r) > 0$ for $r > 0$, $h_i(r) > 0$ for $r \geq 0$, $h_i'(0) = 0$ for $1 \leq i \leq n$.

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It is clear that the isometry group of g contains L.

Let $H = \partial/\partial r$ and $U = f(r)^{-1}v$ for a unit tangent vector v of S^{p-1}. Straightforward calculation yields:

$$\begin{aligned} &\mathrm{Ric}(H,U) = 0, \\ &\mathrm{Ric}(X_i,H) = \mathrm{Ric}(X_i,U) = 0, \quad (1 \le i \le n), \\ &\mathrm{Ric}(X_i,X_j) = 0, \quad (i \ne j, 1 \le i,j \le n). \end{aligned}$$

$$\begin{aligned} \mathrm{Ric}(X_i,X_i) &= -\frac{g_i''}{g_i} - (p-1)\frac{f'g_i'}{fg_i} + \mathrm{Ric}_L(X_i) - \sum_{i\ne j}\frac{g_i'g_j'}{g_ig_j} \\ &\ge \Big\{ -2\alpha_i[(2\alpha_i+1)r^2-1] + (p-1)\alpha_i(2+r^2) \\ &\qquad - c(1+r^2) - \sum_{i\ne j} 4\alpha_i\alpha_j r^2 \Big\}/(1+r^2)^2 \\ &\qquad (1 \le i \le n). \end{aligned} \tag{2}$$

$$\begin{aligned} \mathrm{Ric}(H,H) &= -\sum_{i=1}^{n}\frac{g_i''}{g_i} - (p-1)\frac{f''}{f} \\ &= \left\{ -\sum_{i=1}^{n} 2\alpha_i[(2\alpha_i+1)r^2-1] + (p-1)\frac{r^2+6}{4} \right\}/(1+r^2)^2. \end{aligned} \tag{3}$$

$$\mathrm{Ric}(U,U) = -\frac{f''}{f} + \frac{p-2}{f^2} - (p-2)\left(\frac{f'}{f}\right)^2 - \sum_{i=1}^{n}\frac{f'g_i'}{fg_i}. \tag{4}$$

Since $1-(f')^2 \ge 0$, $f'' \le 0$, we have $\mathrm{Ric}(U,U) > 0$ in (4). Positivity of the Ricci curvature in the equations (2) and (3) follows for p sufficiently large. Observe that every term of the right-hand side decays at a rate of order at least r^{-2}. This completes the proof of the theorem.

REMARK. The smallest p that yields positive Ricci curvature on $M^{p+n} = \mathbf{R}^p \times L$ by means of our construction is quite large in general. For example, in the case of the three-dimensional Heisenberg group $L = H^3$, we have to choose $p > 673$. (With a slightly refined choice of functions, $p > 26$ will already work.) We don't know whether or not p can be chosen much smaller. However, it follows from [**1**] that necessarily $p \ge 4$ when $L = H^3$.

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