BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 19, Number 1, JULY 1988

BI-INVARIANT SCHWARTZ MULTIPLIERS AND LOCAL SOLVABILITY ON NILPOTENT LIE GROUPS

JOE W. JENKINS

Let X denote a finite-dimensional vector space with a fixed positive definite inner product, and let $\mathscr{S}(X)$ denote the Schwartz space on X. We let $\mathscr{MS}(X)$ denote the space of continuous endomorphisms of $\mathscr{S}(X)$ that commute with the action of X on $\mathscr{S}(X)$. The elements of $\mathscr{MS}(X)$ are given by convolution by tempered distributions; i.e., for $E \in \mathscr{MS}(X)$ there is a $D_E \in \mathscr{S}^*(X)$ such that $Ef(x) = \langle D_E, l_x \check{f} \rangle := D_E * f(x)$, where $\check{f}(x) = f(-x)$ and $l_x f(y) = f(y-x)$. Conversely, if $D \in \mathscr{S}^*(X)$, then one can easily see that $E_D \colon f \to D * f$ is a mapping of $\mathscr{S}(X)$ into the smooth functions on X that commutes with translation. Schwartz [S] shows that $E_D \in \mathscr{MS}(X)$ if and only if $\hat{D}$, the Fourier transform of D, is given by a smooth function on X^* which has polynomial bounds on all derivatives. In this note we announce analogues of these results for arbitrary nilpotent Lie groups. Complete proofs will appear elsewhere.

Let N denote a connected, simply connected nilpotent Lie group, with Lie algebra $\mathfrak{n}$. The exponential mapping, $\exp\colon \mathfrak{n} \to N$, is a diffeomorphism, and in terms of the corresponding coordinates left and right translation on N are polynomial mappings. Thus, if $\mathscr{S}(N)$ denotes the image under composition with exp of $\mathscr{S}(\mathfrak{n})$, the right and left action of N on $\mathscr{S}(N)$ are continuous endomorphisms, where $\mathscr{S}(N)$ is topologized so that composition with exp is an isomorphism from $\mathscr{S}(\mathfrak{n})$ to $\mathscr{S}(N)$. We denote by $\mathscr{S}^*(N)$ the dual of $\mathscr{S}(N)$, the space of tempered distributions on N.

For $f \in \mathscr{S}(N)$, the Fourier transform of f, $\hat{f}$, is defined on $\mathfrak{n}^*$, the dual of $\mathfrak{n}$, by

$$\hat{f}(\xi) = \int_{\mathfrak{n}} f(\exp X) e^{-2\pi i \langle \xi, X \rangle}\, dX.$$

One has that $f \to \hat{f}$ is an isomorphism from $\mathscr{S}(N)$ onto $\mathscr{S}(\mathfrak{n}^*)$. For $D \in \mathscr{S}^*(N)$, $\widehat{D}$ is defined on $\mathscr{S}(\mathfrak{n}^*)$ by $\langle \widehat{D}, f \rangle = \langle D, \hat{f} \circ \log \rangle$, where log denotes the inverse of exp.

Let Ad^* denote the coadjoint representation of N on $\mathfrak{n}^*$. A tempered distribution D on $\mathfrak{n}^*$ is said to be Ad^*-invariant if $\langle D, f \circ \mathrm{Ad}^*\, x \rangle = \langle D, f \rangle$ for all $x \in N$ and $f \in \mathscr{S}(\mathfrak{n}^*)$. A tempered distribution D on N is said to be bi-invariant if $\langle D, \mathfrak{r}_{x^{-1}} f \rangle = \langle D, l_x f \rangle$ for all $f \in \mathscr{S}(N)$, where $\mathfrak{r}_x f(y) = f(yx)$ and $l_x f(y) = f(x^{-1} y)$ for all $x, y \in N$. A straightforward computation shows that an element $D \in \mathscr{S}(N)$ is bi-invariant if and only if $\widehat{D}$ is Ad^*-invariant.

Received by the editors January 22, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 22E30, 43A55.
This research supported in part by a grant from the National Science Foundation.

Let $\mathscr{MS}(N)$ denote the space of continuous endomorphisms on $\mathscr{S}(N)$ that commute with both right and left translations by elements of N. As in the Euclidean case, one has that for each $E \in \mathscr{MS}(N)$ there is a $D_E \in \mathscr{S}^*(N)$ such that $Ef = D_E * f$, where, as before, $D_E * f(x) := \langle D_E, l_x \check{f} \rangle$. If $D \in \mathscr{S}^*(N)$ we denote by E_D the mapping defined on $\mathscr{S}(N)$ by $E_D f = D * f$.

Let $PB_N^\infty(\mathfrak{n}^*)$ denote the space of smooth, Ad^*-invariant functions defined on $\mathfrak{n}^*$ with polynomial bounds on all derivatives. This space is topologized using the seminorms ν_{ij} defined on $PB_N^\infty(\mathfrak{n}^*)$ by

$$\nu_{ij}(\theta) = \sup_{|\alpha| \leq j} \sup_{\xi \in \mathfrak{n}^*} |\partial^\alpha \theta(\xi)|/(1 + ||\xi||^2)^i,$$

where ∂^α denotes the standard differential operator corresponding to the multi-index α, and some fixed basis of $\mathfrak{n}^*$. A sequence $\{E_n\} \subset \mathscr{MS}(N)$ converges to 0 if $E_n f \to 0$ in $\mathscr{S}(N)$ for each $f \in \mathscr{S}(N)$.

THEOREM A. *The mapping $\mathscr{MS}(N) \to PB_N^\infty(\mathfrak{n}^*)$: $E \to \widehat{D}_E$ is a homeomorphism and an algebra isomorphism, the products being composition on $\mathscr{MS}(N)$ and pointwise multiplication on $PB_N^\infty(\mathfrak{n}^*)$.*

For $\xi \in \mathfrak{n}^*$, let π_ξ denote the irreducible unitary representation of N that corresponds to the Ad^*-orbit of ξ by the Kirillov theory. For $\theta \in PB_N^\infty(\mathfrak{n}^*)$, let D_θ be the tempered distribution on N with Fourier transform θ.

THEOREM B. *For $\theta \in PB_N^\infty(\mathfrak{n}^*)$, $f \in \mathscr{S}(N)$, and $\xi \in \mathfrak{n}^*$,*

$$\pi_\xi(D_\theta * f) = \theta(\xi)\pi_\xi(f).$$

As an application of these results, we consider the question of local solvability. Recall that a left invariant differential operator L on N is said to be locally solvable if there is an open set $U \subset N$ such that $C_c^\infty(U) \subset L(C^\infty(U))$.

Let $o(\xi)$ denote the Ad^*-orbit in $\mathfrak{n}^*$ that contains ξ, and having fixed a norm on $\mathfrak{n}^*$, set $|o(\xi)| = \inf\{||\xi'||: \xi' \in o(\xi)\}$. Suppose that N contains a discrete, cocompact subgroup Γ. Then $L^2(\Gamma\backslash N)$ is a direct sum of subspaces $\mathscr{H}_\xi$ such that the restriction to $\mathscr{H}_\xi$ of right translation is a finite multiple of π_ξ. We denote by $(\Gamma\backslash N)_0^\wedge$ the elements of $\widehat{N}$ appearing in this decomposition that are in general position.

THEOREM C. *Let L be a left invariant differential operator on N. Suppose that for each $\pi_\xi \in (\Gamma\backslash N)_0^\wedge$, $\pi_\xi(L)$ has a bounded right inverse A_ξ on $\mathscr{H}_\xi$, and that the norm of A_ξ is bounded by a polynomial in $|o(\xi)|$. Then L is locally solvable.*

The proof of Theorem A requires the introduction of somewhat more general spaces. Let $\mathfrak{h}$ be a subspace of the center of $\mathfrak{n}$, and let $\lambda \in \mathfrak{h}^*$. We define the unitary character χ_λ on $H := \exp(\mathfrak{h})$ by $\chi_\lambda(\exp X) = e^{2\pi i \langle \lambda, X \rangle}$, and denote by $\mathscr{S}(N/H, \chi_\lambda)$ the space of all smooth functions f defined on N such that $f(xy) = \chi_\lambda(y) f(x)$ for all $x \in N$, $y \in H$, and such that $f \circ \exp_{|\mathfrak{k}} \in \mathscr{S}(\mathfrak{k})$, where $\mathfrak{k}$ is a complement to $\mathfrak{h}$ in $\mathfrak{n}$. The topology of $\mathscr{S}(N/H, \chi_\lambda)$ is defined by requiring that the mapping $f \to f \circ \exp_{|\mathfrak{k}}$ be a homeomorphism. Define $P_\lambda: \mathscr{S}(N) \to \mathscr{S}(N/H, \chi_\lambda)$ by

$$P_\lambda f(\exp X) = \int_{\mathfrak{h}} f(\exp(X + Y))\chi_\lambda(-Y)\, dY.$$

P_λ is an open surjection and thus its adjoint P_λ^* is an isomorphism of $\mathscr{S}^*(N/H, \chi_\lambda)$ into $\mathscr{S}^*(N)$.

Let $\mathfrak{h}^\perp$ be the annihilator of $\mathfrak{h}$ in $\mathfrak{n}^*$. For $\lambda \in \mathfrak{h}^*$ (identified as a subspace of $\mathfrak{n}^*$), there is a natural Schwartz space on $\mathfrak{h}^\perp + \lambda$, $\mathscr{S}(\mathfrak{h}^\perp + \lambda)$, given by composing elements of $\mathscr{S}(\mathfrak{h}^\perp)$ with translation by $-\lambda$. Considering $\mathscr{S}(N/H, \chi_\lambda)$ and $\mathscr{S}(\mathfrak{h}^\perp + \lambda)$ as subspaces of $\mathscr{S}^*(N)$ and $\mathscr{S}^*(\mathfrak{n}^*)$ respectively, the Fourier transform is defined on these spaces and one has that $f \to \hat{f}$ is an isomorphism of $\mathscr{S}(N/H, \chi_\lambda)$ onto $\mathscr{S}(\mathfrak{h}^\perp + \lambda)$ and of $\mathscr{S}(\mathfrak{h}^\perp + \lambda)$ onto $\mathscr{S}(N/H, \chi_{-\lambda})$. Also one has that for $D \in \mathscr{S}^*(N/H, \chi_\lambda)$, $(P_\lambda^* D)^\wedge = R_{-\lambda}^* \widetilde{D}$, where $R_\lambda \colon \mathscr{S}(\mathfrak{n}^*) \to \mathscr{S}(\mathfrak{h}^\perp + \lambda)$ is restriction, and $\widetilde{D}$ is the element in $\mathscr{S}^*(\mathfrak{h}^\perp - \lambda)$ defined by $\langle \widetilde{D}, f \rangle = \langle D, \hat{f} \rangle$. Thus $(P_\lambda^* D)^\wedge$ is supported on $\mathfrak{h}^\perp + \lambda$ and has no normal derivatives.

For $f \in \mathscr{S}(N/H, \chi_\lambda)$ and $D \in \mathscr{S}^*(N/H, \chi_{-\lambda})$, the convolution $D * f$ is defined by setting $D * f(x) = \langle D, l_x(\check{f}) \rangle$ for each $x \in N$. Suppose now that $D \in \mathscr{S}^*(N)$ and $f \in \mathscr{S}(N)$. One can use Abelian Fourier analysis to study the mapping defined on $\mathfrak{z}$, the center of $\mathfrak{n}$, by $Y \to D * f(\exp(X + Y))$. If this mapping is in $\mathscr{S}(\mathfrak{z})$, then

$$D * f(\exp X) = \int_{\mathfrak{z}^*} P_\lambda(D * f)(\exp X)\, d\lambda,$$

for appropriately normalized Lebesgue measure $d\lambda$. Furthermore, $P_\lambda(D * f) = D_\lambda * P_\lambda f$, where D_λ is the element of $\mathscr{S}^*(N/H, \chi_{-\lambda})$ whose Fourier transform, $\widetilde{D}_\lambda$, agrees with the restriction to $\mathfrak{h}^\perp + \lambda$ of $\widehat{D}$. Thus, convolution between elements of $\mathscr{S}^*(N)$ and $\mathscr{S}(N)$ decomposes into convolutions between elements of $\mathscr{S}^*(N/H, \chi_{-\lambda})$ and $\mathscr{S}(N/H, \chi_\lambda)$ in such a way that smoothness and growth conditions on $\widehat{D}$, $D \in \mathscr{S}^*(N)$ are inherited by $\widetilde{D}_\lambda$, $D_\lambda \in \mathscr{S}^*(N/H, \chi_{-\lambda})$. One then proceeds by induction on the dimension of N/H. Of course, this requires maintaining considerable control of the various seminorm estimates that appear in the decompositions.

The proof of Theorem B follows along the usual induction argument lines with the Plancherel Theorem being used to reduce the dimension.

For Theorem C, one constructs a θ on $\mathfrak{n}^*$ such that both θ and $1/\theta$ are in $PB_N^\infty(\mathfrak{n}^*)$, and such that $\sum \|A(\xi)\| |\theta(\xi)| < \infty$, the sum being over $(\Gamma \backslash N)_0^\wedge$. One then uses the fact that $(D_{1/\theta} * f) * (D_\theta * g) = f * g$ and the Dixmier and Mallivan [**DM**] factorization to complete the proof.

REMARKS. The fact that $D_\theta \in \mathscr{MS}(N)$ was proved by R. Howe in [**H**], and indeed, the ideas presented there are the foundation of this work. Theorem B was proved for the case where θ is a polynomial by A. Kirillov in [**K**]. In [**CG**], L. Corwin and F. Greenleaf proved Theorem C with the additional assumption that all the representations in general position were induced from a common, normal subgroup. One-sided Schwartz multipliers have been studied by L. Corwin in [**C**].

REFERENCES

[**C**] L. Corwin, *Tempered distributions on Heisenberg groups whose convolution with Schwartz class functions is Schwartz class*, J. Funct. Anal. **44** (1981), 328–347.

[**CG**] L. Corwin and F. Greenleaf, *Solvability of certain left-invariant differential operators by nilmanifold theory*, Comm. Pure Appl. Math. **36** (1983), 755–765.

[**DM**] J. Dixmier et P. Mallivan, *Factorisations de fonctions et de vecteurs indéfiniment différentiables*, Bull. Sci. Math. **102** (1978), 305–330.

[**H**] R. Howe, *On a connection between nilpotent groups and oscillatory integrals associated to singularities*, Pacific J. Math. **73** (1977), 329–364.

[**K**] A. Kirillov, *Unitary representations of nilpotent Lie groups*, Uspekhi Mat. Nauk. **17** (1962), 57–110.

[**R**] M. Rais, *Solutions elémentaires des operateurs différentiels bi-invariants sur un groupe de Lie nilpotent*, C. R. Acad. Sci. Paris Sér. A-B **273** (1971), A495–498.

[**S**] L. Schwartz, *Théorie des distributions*, tome II, Hermann, Paris, 1959.

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY AT ALBANY, STATE UNIVERSITY OF NEW YORK, ALBANY, NEW YORK 12222