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CONSTRAINED POISSON ALGEBRAS AND STRONG HOMOTOPY REPRESENTATIONS

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A Poisson algebra is a commutative associative algebra A with an (anticommutative) bracket $\{\ ,\ \}$ which is a derivation with respect to the commutative product: $\{f, gh\} = \{f, g\}h + f\{g, h\}$. Constraints constitute a distinguished set of elements ϕ_α of A. They are said to be *first class* constraints if the ideal I they generate (under the commutative product) is closed under Poisson bracket; I need not be an ideal with respect to $\{\ ,\ \}$. This structure arises in physics with $A = C^\infty(W)$ for some symplectic manifold W. The constraints determine a subvariety $V \subset W$, the zero locus of I, and a foliation $\mathscr{F}$ of V, by the flows determined by the derivations $\{\ ,\ \}$. One wishes to compute the $\operatorname{ad} I$-invariant functions on V, which would give $C^\infty(V/\mathscr{F})$ were the foliation to give a submersion $V \to V/\mathscr{F}$ onto a manifold.

In a remarkable series of papers, Fradkin, Batalin and Vilkovisky [**0**–**3**, **6**] and then Henneaux [**10**] developed a method for calculating the $\operatorname{ad} I$-invariant functions in $C^\infty(V) = A/I$ without passing through the quotient A/I. The method appeared to depend on solving certain specific, complicated equations and initially was applicable only locally and when I was a regular ideal.

Using the techniques of 'homological perturbation theory' [**7**, **8**, **9**], I am able to justify their machinery in terms of the algebra alone, including, with Henneaux [**11**], the case of nonregular ideals [**0**]. The idea for this approach owes a great deal to the paper of Browning and McMullan [**4**], which revealed the structure of a multicomplex implicit in Fradkin et al and Henneaux.

The Lie algebra cohomology $H^0(I, A/I)$ computes the $\operatorname{ad} I$-invariant functions on V, but physics requires a description in terms of A and prefers to use Φ, the linear span of the constraints ϕ_α, rather than the full ideal I. An obvious step algebraically is to replace A/I by a free resolution over A. To combine this with the restriction to $\Phi \subset I$ is more subtle.

The Lie algebra cohomology of Cartan, Chevalley and Eilenberg [**5**] begins with the algebra $\operatorname{Alt}(I, A/I)$ of alternating multilinear functions on I with values in A/I and a differential $\operatorname{Alt} \to \operatorname{Alt}$ (which increases the number of variables by one) given in terms of the bracket on I and the adjoint representation of I on A/I: For example, for $h: I \to A$, we have

$$(\delta h)(f, g) = h(\{f, g\}) - \{f, h(g)\} + \{g, h(f)\}.$$

The subalgebra $\operatorname{Alt}_A(I, A/I)$ of A-multilinear functions is in fact a subcomplex with the same H^0. (This is isomorphic to the complex which defines

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the Rinehart cohomology of the $(A/I, R)$-Lie algebra I/I^2 with coefficients in A/I [**12**].) The inclusion $\Phi \subset I$ induces $\mathrm{Alt}_A(I, A/I) \to \mathrm{Alt}(\Phi, A/I)$ and a differential also denoted δ. (This map is an isomorphism if I is regular.)

Now introduce a multiplicative resolution $\pi \colon K_I \to A/I$, that is, K_I is a graded commutative differential algebra (with differential d) and π induces an isomorphism $\pi^* \colon H_0(K_I) \to A/I$ with $H_i(K_I) = 0$ otherwise. For example, if I is a regular ideal, take K_I to be the Koszul complex; more generally, the Tate resolution will do [**14**]. If we replace A/I by K_I and consider $\mathrm{Alt}(\Phi, K_I)$, the problem is to extend d to a differential D so as to realize the same homology as that of $\mathrm{Alt}(\Phi, A/I)$ with respect to δ. The major source of difficulty is that the adjoint representation of I on A/I does *not* lift to K_I; in spite of this, we have:

THEOREM 1. *There are differentials δ_i on $\mathrm{Alt}(\Phi, K_I)$ which increase the form degree by i and the resolution degree by $i-1$ such that $\delta_0 = d$ and $D = \sum \delta_i$ has $D^2 = 0$ with $\pi \colon K_I \to A/I$ inducing*

$$H^0(\mathrm{Alt}(\Phi, K_I), D) \approx H^0(\mathrm{Alt}(I, A/I), \delta).$$

Our proof of the theorem uses the methods of homological perturbation theory [**7**, **8**, **9**]. Let $\mathrm{Der}^q K$ denote the derivations of K_I which increase resolution degree by q. The collection $\mathrm{Der}\, K_I = \{\mathrm{Der}^q K_I\}$ is made into a differential graded Lie algebra by using the graded commutator of derivations and the induced differential: $d\theta = [d, \theta]$. We cannot, in general, find a representation of I in $\mathrm{Der}\, K_I$, but we can find a "strong homotopy representation", meaning a family $\Theta_i \in \mathrm{Alt}^i(I, \mathrm{Der}\, K_I)$ for $i \geq 1$ satisfying the following relations: For $i = 1$, $\Theta^1(f) = \{f, \ \}$. For $i > 1$, and $\bar{f}_i = (f_0, \dots, f_i)$,

$$(*) \quad [d, \Theta^i]\bar{f}_i = \sum [\Theta^j, \Theta^k](\bar{f}_i) + \sum (-1)^{j+k} \Theta^{i-1}(\{f_j, f_k\}, \dots, \hat{j}, \dots, \hat{k}, \dots).$$

For $i = 1$, this is to be interpreted as $[d, \theta^1] = 0$. Here $[\ ,\ ]$ is the usual induced bracket on $\mathrm{Alt}(V, L)$ for a vector space V and Lie algebra L. The maps Θ^i are constructed inductively, using a contracting homotopy s for K_I, that is: $sd + ds = 1 - \bar{\pi}$ where $\bar{\pi} \colon K_I \to A/I \to A \hookrightarrow K_I$ and the map $1 - \bar{\pi}$ is the identity on I. We begin by defining $\Theta^1 \colon I \to \mathrm{Der}^0 K_I$ as an extension of the adjoint action of I on A as follows: By induction on the resolution degree of a generator x of K_I over A, define $\Theta^1(f)(x) = s\Theta^1(f)(dx)$. Verify directly that $(*)$ is valid in the form $[d, \Theta^1] = 0$. Now assume we have constructed Θ^i for $i < n$ to satisfy $(*)$. Let RHS denote the right-hand side of the equation $(*)$ for $i = n$. Verify that $[d, \mathrm{RHS}] = 0$ using $(*)$ and the Jacobi identity. Now define the derivation $\Theta^n(\bar{f}_n)$ as sRHS. We verify that

$$\begin{aligned} [d, s\mathrm{RHS}] &= ds\mathrm{RHS} + s\mathrm{RHS}d = (ds + sd)\mathrm{RHS} \quad \text{by induction} \\ &= (1 - \bar{\pi})\mathrm{RHS} = \mathrm{RHS}, \end{aligned}$$

since RHS raises resolution degree by at least $j - 1 + k - 1$, which is into the kernel of π unless $j = k = 1$. For $n = 2$, we also use the fact that Θ^1 is an extension of the adjoint action of I on A in terms of the original Poisson bracket.

Since $\Phi \subset I$ need not be closed under the bracket, we cannot just restrict D to $\mathrm{Alt}(\Phi, K_I)$. Instead, the FBV construction in the regular case makes further use of the Poisson algebra. Notice that the Koszul resolution can be written as $A \otimes \bigwedge s\Phi$ where $s\Phi$ is isomorphic to Φ as a vector space, while $\mathrm{Alt}(\Phi, K_I)$ contains the vector space dual $\Phi^* = \mathrm{Hom}(\Phi, R)$. Extend the Poisson bracket of A to all of $\mathrm{Alt}(\Phi, K_I)$ by first defining $\{\Phi^*, s\Phi\}$ to be isomorphic to the usual dual pairing and then extending to a graded Poisson bracket by using the derivation property: $\{\omega, \eta \wedge \varsigma\} = \{\omega, \eta\} \wedge \varsigma + (-1)^{|\omega||\eta|} \eta \wedge \{\omega, \varsigma\}$.

THEOREM 2. *There is an element* $Q \in \prod \mathrm{Alt}^p(\Phi, K_I)$ *such that* D *in Theorem* 1 *is given by* $D = \{Q, \ \}$.

We write $Q = \sum Q_p$ where $Q_p \in \mathrm{Alt}^{p+1}(\Phi, K_I)$ takes values in $A \otimes \bigwedge^p s\Phi$. Although $D = \{Q, \ \}$, we do *not* have $\delta_i = \{Q_i, \ \}$ but rather δ_i is of bidegree $(i, i-1)$, while $\{Q, \ \}$ has components of bidegree $(i, i-1)$ and $(i+1, i)$. To start, let Q_0 be the inclusion $\iota\colon \Phi \hookrightarrow A \hookrightarrow K_I$ so that $\{Q_0, \ \}|K_I$ is the Koszul differential. (This is easier to see in terms of a basis $\{\phi_\alpha\}$ for Φ, dual basis $\{\eta^\alpha\}$ for Φ^*, and basis $\{\mathscr{P}_\alpha\}$ for $s\Phi$ so that $Q_0 = \phi_\alpha \eta^\alpha$.) Filter $\mathrm{Alt}(\Phi, K_I)$ by $F^p = \sum_{i \le p} \mathrm{Alt}^i$, and for any element R of the complex, let R^2 denote $\frac{1}{2}\{R, R\}$. Now construct Q_i by induction so that the partial sums $R_i = \sum Q_j$ have the following properties:

$$R_p^2 \in F^{p+2} \quad \text{and} \quad dR_p^2 \in F^{p+3}.$$

Define $Q_{n+1} = -sR_n^2$. A slightly complicated computation then shows that R_{n+1} satisfies the inductive hypothesis.

We have left to show that D gives the desired homology. The resolution $\pi\colon K_I \to A/I$ induces a map of complexes. If we filter $\mathrm{Alt}(\Phi, K_I)$ as above, the associated graded has differential just d with homology $\mathrm{Alt}(\Phi, A/I)$. A standard spectral sequence argument then gives the desired result.

Because of the motivating physics, Fradkin et al consider also the situation in which A is a super-Poisson algebra, i.e. $\mathbf{Z}/2$-graded with appropriate signs throughout. Now we need to use a super-resolution, for example, Jozefiak's [**13**]. The formalism we have used need only be made super (i.e. attend carefully to signs) with some extra care interpreting formal power series.

As a guide to the physics literature, in the regular case, Q_i corresponds to an expression $U^{\bar\alpha}_{\underline\beta} \eta^{\underline\beta} \mathscr{P}_{\bar\alpha}$ where $\bar\alpha = \alpha_1 \cdots \alpha_{i-1}$, $\underline\beta = \beta_1 \cdots \beta_{i+1}$ and $\eta^{\underline\beta} = \eta^{\beta_1} \wedge \cdots \wedge \eta^{\beta_{i+1}}$, etc. Finally, the η^β are called ghosts, the $\mathscr{P}_\alpha$ anti-ghosts and, in the nonregular case, syzygies are called extraghosts or ghosts-of-ghosts-of-....

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