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STRUCTURE OF FOURIER AND FOURIER–STIELTJES COEFFICIENTS OF SERIES WITH SLOWLY VARYING CONVERGENCE MODULI

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Ostensibly, convergence problems regarding the Fourier series in L^1 parallel the classical Tauberian problems. Let $f \in L^1(T)$, $T = \mathbf{R}/2\pi\mathbf{Z}$, then the partial sums $S_n(f) = S_n(f,t) = \sum_{|k|\le n} \hat{f}(k)e^{ikt}$ are $(C,1)$-summable, both pointwise and in L^1-norm. Inasmuch as the appropriate Tauberian conditions are available, the convergence questions may be settled in the standard manner. However, Tauberian conditions needed to recover L^1-convergence originate from the Hausdorff-Young inequality and do not have a straightforward analogue in the elementary Tauberian theory. Such a condition is obtained in [**1**], i.e.

$$\lim_{\lambda\to 1+0} \overline{\lim_{n}} \sum_{|k|=n+1}^{[\lambda n]} |k|^{p-1}|\Delta \hat{f}(k)|^p = 0, \tag{1}$$

where $1 < p \le 2$ and $f \in L^1(T)$. Later in [**2** and **3**], the condition (1) has been further extended and studied. Although (1) is much weaker than the classical [**4**, **5**] and neoclassical [**6**, **7**] regularity and/or speed conditions, it does not provide explicit information about the Fourier coefficients. To overcome this shortcoming a new approach is proposed in [**8**], based on regular variation of the convergence moduli.

A nondecreasing sequence $\{R(n)\}$ of positive numbers is $*$-regularly varying if

$$\lim_{\lambda\to 1+0} \overline{\lim_{n}} \frac{R([\lambda n])}{R(n)} \le 1;$$

or more generally, the sequence $\{R(n)\}$ is O-regularly varying if

$$\overline{\lim_{n}} \frac{R([\lambda n])}{R(n)}$$

is finite for $\lambda > 1$. In particular, if $\lim_n R([\lambda n])/R(n) = 1$, $\{R(n)\}$ is slowly varying.

Let $\{c(n)\}$ be a sequence of complex numbers and let $\sum_{|n|<\infty} c(n)e^{int}$ be its formal trigonometric transform. The convergence modulo of the trigonometric transform is defined as

$$K_n^p(c) = \sum_{|k|\le n} |k|^{p-1}|\Delta c(k)|^p, \qquad p > 1.$$

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The following two theorems are proved in [**8**]. In the second theorem $M(T)$ denotes the class of all complex Borel measures on T.

THEOREM A. *Let $f \in L^1(T)$ and for some $1 < p \le 2$ and some $*$-regularly varying sequence $\{R(n)\}$ let $K_n^p(f) = \log R(n)$. Then the Fourier series of f converges a.e. and it converges in $L^1(T)$-norm if and only if $\widehat{f}(n)\log|n| = o(1)$, $|n| \to \infty$.*

THEOREM B. *Let $\mu \in M(T)$ and let $(\widehat{\mu}(n) - \widehat{\mu}(-n))\log n = O(1)$, $n \to \infty$. If for some $1 < p \le 2$ and some O-regularly varying sequence $\{R(n)\}$, $K_n^p(\mu) = \log R(n)$, then*

$$\|S_n(\mu)\| = B_n|\widehat{\mu}(n)|\log n + O(1), \qquad n \to \infty,$$

where $\{B_n\}$ is a bounded sequence of real numbers, bounded away from zero.

This theorem is a quantitative version of Helson's [**9**] theorem, and it is given here in somewhat different form than in [**8**] to match the form of Theorem A.

In this announcement we shall show that: (i) $*$-regular variation in Theorem A can be lightened to O-regular variation; and (ii) there is a representation for $\{\widehat{f}(n)\}$ and $\{\widehat{\mu}(n)\}$ in terms of Fourier coefficients of functions in $L^q(T)$, $1/p + 1/q = 1$. Consequently, Theorem A and Theorem B can be unified in a new theorem that improves both theorems, in particular Theorem A.

THEOREM. *Let $\{c(n)\}_{|n|<\infty}$ be a sequence of complex numbers and for some $1 < p \le 2$ and some O-regularly varying sequence $\{R(n)\}$ let*

$$K_n^p(c) = \log R(n). \tag{2}$$

Then

(i) *there exists $h \in L^q(T)$, $1/p + 1/q = 1$, such that*

$$c(k) = \frac{1+\operatorname{sgn}(k)}{2}c(0) + \frac{1-\operatorname{sgn}(k)}{2}c(-1) - \sum_{j=(1-\operatorname{sgn}(k))/2}^{|k|-1} \widehat{h}(j\operatorname{sgn}(k)), \quad k \in \mathbf{Z};$$

(ii) *for $c(n) = o(1)$, $|n| \to \infty$, the trigonometric transform of $\{c(n)\}$ converges a.e. and $c(k) = \sum_{j=|k|}^{\infty} \widehat{h}(j\operatorname{sgn}(k))$, $k \in \mathbf{Z}$;*

(iii) *for $c = \widehat{f}$, $f \in L^1(T)$, the Fourier series of f converges in $L^1(T)$-norm if and only if $\widehat{f}(n)\log|n| = o(1)$, $|n| \to \infty$.*

(iv) *for $c = \widehat{\mu}$, $\mu \in M(T)$, and $(\widehat{\mu}(n) - \widehat{\mu}(-n))\log n = O(1)$, $n \to \infty$;*

$$\|S_n(\mu)\| = B_n|\widehat{\mu}(n)|\log n + O(1), \qquad n \to \infty,$$

where $\{B_n\}$ is a bounded sequence of real numbers, bounded away from zero.

OUTLINE OF PROOF. The case $K_n^p(c) = O(1)$, $n \to \infty$, is trivial. Assume $K_n^p(c) \to \infty$, $n \to \infty$. Then condition (2) is equivalent to:

$$\varlimsup_{n} \sum_{|k|=n+1}^{[\lambda n]} |k|^{p-1}|\Delta c(k)|^p \quad \text{is finite for } \lambda > 1.$$

Hence $\{K_n^p(c)\}$ is slowly varying. Also

$$\sum_{|j|\leq n} |\Delta c(j)|^p \leq C_1 \sum_{|j|\leq n} \frac{K_j^p(c)}{|j|^p} + C_2 \frac{K_n^p(c)}{|n|^{p-1}}, \tag{3}$$

where C_1 and C_2 are absolute constants. The series on the left-hand side of (3) is therefore convergent. By F. Riesz's theorem [**10**] there is an $h \in L^q(T)$, $1/p + 1/q = 1$, such that

$$h(t) = \sum_{|n|<\infty} \Delta c(n) e^{int}, \tag{4}$$

and the representation in (i) follows.

The series (4) converges a.e. by L. Carleson's theorem [**11**], for $c(n) = o(1)$, $|n| \to \infty$, and for $t \neq 0$, $\sum_{|n|<\infty} c(n)e^{int}$ converges a.e. The representation in (ii) is now evident.

The statement (iii) is a considerable generalization of Theorem 2.1 in [**8**] and the proof requires several refinements. Let $\lambda > 1$. Define

$$T_n(\lambda) = (-\pi, -\pi/(\lambda-1)n) \cup (\pi/(\lambda-1)n, \pi)$$

and

$$\tau_n(f,\lambda) = \tau_n(f,t,\lambda) = \frac{1}{[\lambda n]-n} \sum_{k=n+1}^{[\lambda n]} S_k(f).$$

Then

$$\begin{aligned} \|\tau_n(f,\lambda) - S_n(t)\|_{L^1(t)} &= \int_{T_n(g)} |\tau_n(f,t,\lambda) - S_n(f,t)|\,dt \\ &\quad + \int_{T-T_n(\lambda)} |\tau_n(f,t,\lambda) - S_n(f,t)|\,dt \\ &= J_1 + J_2. \end{aligned}$$

For J_2 we have the uniform estimates

$$J_2 \leq \frac{1}{[\lambda n]-n} \sum_{|k|=n+1}^{[\lambda n]} |\hat{f}(k)| = o(1), \qquad n \to \infty.$$

The integral J_1 can be written as a sum of two integrals J_{11} and J_{12}. Applying Hölder's inequality to J_{11}, followed by the Hausdorff-Young inequality, we get

$$J_{11} \leq C_3(\lambda-1)^{1/q} \sum_{|k|=n+1}^{[\lambda n]} |k|^{p-1} |\Delta \hat{f}(k)|^p,$$

where C_3 is an absolute constant. Thus

$$\lim_{\lambda\to 1+0} \overline{\lim_{n}}\, J_{11} = 0.$$

Therefore

$$\lim_{\lambda\to 1+0} \overline{\lim_{n}}\, \|\tau_n(f,\lambda) - S_n(f)\|_{L^1(T)} = 0 \tag{4}$$

is equivalent to

$$\lim_{\lambda\to 1+0} \overline{\lim_{n}}\, J_{12} = 0.$$

However

$$J_{12} = \int_{T_n(\lambda)} \left| \frac{\hat{f}(n)e^{int} + \hat{f}(-n)e^{-int}}{1 - e^{-it}} \right| dt,$$

and $\overline{\lim}_n J_{12} = 0$ if and only if $\hat{f}(n)\log|n| = o(1)$, $|n| \to \infty$. Hence (4) is equivalent to $\hat{f}(n)\log|n| = o(1)$, $|n| \to \infty$. A standard argument completes the proof.

The proof of (iv) follows the lines of the proof of Theorem 2.3 in [**8**].

The details and proofs will appear elsewhere.

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