

TOPOLOGICAL RIGIDITY FOR HYPERBOLIC MANIFOLDS

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ABSTRACT. Let M be a complete Riemannian manifold having constant sectional curvature -1 and finite volume. Let \overline{M} denote its Gromov-Margulis manifold compactification and assume that the dimension of M is greater than 5. (If M is compact, then $\overline{M} = M$ and $\partial\overline{M}$ is empty.) We announce (among other results) that any homotopy equivalence $h: (N, \partial N) \rightarrow (\overline{M}, \partial\overline{M})$, where N is a compact manifold, is homotopic to a homeomorphism. This is a topological analogue of Mostow's rigidity theorem [18]. Moreover, for each integer j , the surgery group $L_j(\pi_1 M)$ is isomorphic to the set of homotopy classes of maps $[I^k \times \overline{M} \text{ rel } \partial, G/\text{Top}]$ where k is any positive integer such that $k + \dim M \equiv j \pmod{4}$. Here I^k denotes the k -fold product $I \times I \times \cdots \times I$ where I is the closed interval $[0, 1]$.

Let M denote a complete Riemannian manifold having constant sectional curvature -1 and finite volume. Thus M is a real hyperbolic manifold of finite volume. Gromov [13] and Margulis have constructed a smooth manifold compactification of M which is denoted by \overline{M} . Let I^k denote the k -fold product $I \times I \times \cdots \times I$, where I is the closed interval $[0, 1]$; in particular, I^0 is a single point. Let N be a compact manifold such that its boundary ∂N decomposes as $\partial N = \partial_1 N \cup \partial_2 N$ where $\partial_1 N, \partial_2 N$ are compact codimension zero submanifolds of ∂N with $\partial(\partial_1 N) = \partial(\partial_2 N) = \partial_1 N \cap \partial_2 N$. Set $\Lambda N = \partial(\partial_1 N)$.

THEOREM 1. *Let $h: (N, \partial_1 N, \partial_2 N, \Lambda N) \rightarrow (I^k \times \overline{M}, \partial I^k \times \overline{M}, I^k \times \partial\overline{M}, \partial I^k \times \partial\overline{M})$ be a homotopy equivalence of 4-tuples such that $h: \partial_1 N \rightarrow (\partial I^k \times \overline{M})$ is a homeomorphism. If $k + \dim(M) > 5$, then there is a homotopy*

$$h_t: (N, \partial_1 N, \partial_2 N, \Lambda N) \rightarrow (I^k \times \overline{M}, \partial I^k \times \overline{M}, I^k \times \partial\overline{M}, \partial I^k \times \partial\overline{M}), \quad t \in [0, 1],$$

with $h_0 = h$, h_1 a homeomorphism and the restriction of h_t to $\partial_1 N$ the constant homotopy. Moreover, if the restriction of h to $\partial_2 N$ is also a homeomorphism, then we need only assume that $k + \dim(M) > 4$ and h_t can be constructed so that it is constant on all of ∂N .

COROLLARY 1. *Let $h: (N, \partial N) \rightarrow (\overline{M}, \partial\overline{M})$ be a homotopy equivalence of pairs where N is a compact manifold. If $\dim(M) > 5$, then there is a homotopy*

$$h_t: (N, \partial N) \rightarrow (\overline{M}, \partial\overline{M}), \quad t \in [0, 1],$$

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with $h_0 = h$ and such that h_1 is a homeomorphism. Moreover, if $h: \partial N \rightarrow \partial \overline{M}$ is already a homeomorphism, then we need only assume that $\dim(M) > 4$ and h_t can be constructed so that it is constant on ∂N .

Recall that each component of $\partial \overline{M}$ is homeomorphic to a flat Riemannian manifold. Hence [7] can be used to reduce the proof of Theorem 1 to the case where $h: \partial_2 N \rightarrow I^k \times \partial \overline{M}$ is a homeomorphism.

Let $w_1: \pi_1(M) \rightarrow \mathbf{Z}_2$ denote the homomorphism determined by the first Stiefel-Whitney class of M , and $L_j(\pi_1 M)$ denote the j -dimensional surgery group for the fundamental group $\pi_1 M$ with orientation data w_1 , defined by Wall in [21]. Let

$$\Theta: [I^k \times \overline{M} \text{ rel } \partial, G/\text{Top}] \rightarrow L_{k+m}(\pi_1 M)$$

be the surgery homomorphism where $m = \dim M$ (cf. [21 and 17]).

THEOREM 2. *For each positive integer k , the surgery homomorphism*

$$\Theta: [I^k \times \overline{M} \text{ rel } \partial, G/\text{Top}] \rightarrow L_{k+m}(\pi_1 M)$$

is an isomorphism provided $k + m > 5$.

It was shown in [5 and 6] that Θ is a split monomorphism. Since $L_j(\pi_1 M)$ is isomorphic to $L_{j+4}(\pi_1 M)$ for all integers j , Theorem 2 yields a calculation of $L_j(\pi_1 M)$ regardless of the dimension of M ; in particular, even when M is either three- or four-dimensional. Since G/Top has the same rational homotopy type as does the product $\prod_{j \geq 1} K(\mathbf{Z}; 4j)$ of Eilenberg-Mac Lane spaces, Theorem 2 has the following consequence.

COROLLARY 2. *The surgery group $L_{k+m}(\pi_1 M) \otimes \mathbf{Q}$ is isomorphic to the direct sum of cohomology groups*

$$\bigoplus_{j=-\infty}^{\infty} H^{4j-k}(\overline{M}, \partial \overline{M}; \mathbf{Q})$$

where $m = \dim M$ and $H^i(\overline{M}, \partial \overline{M}; \mathbf{Q})$ denotes the i th cohomology group of the pair $(\overline{M}, \partial \overline{M})$ with (untwisted) rational coefficients.

The techniques used to prove Theorem 2 may also be applied to compute the surgery groups of other fundamental groups. For example, let G be a finitely presented group, $\Gamma = G \times \pi_1(M)$ and $w_1: \Gamma \rightarrow \mathbf{Z}_2$ be a homomorphism which restricts to the identically zero homomorphism on $\pi_1(M)$; let $L_j(\Gamma)$ and $L_j(G)$ denote the surgery groups for the groups Γ and G with orientation data w_1 and w_1 restricted to G ; respectively, and assume that M is orientable. Let $K_{\mathbf{F}/\text{TOP}}^*(\)$ denote the cohomology theory defined in [16].

THEOREM 3. *Suppose that $\text{Wh}(G \times C^n) = 0$ holds for all nonnegative integers n where $\text{Wh}(\)$ is the Whitehead group and C^n is the free abelian group of rank n . Then, for each integer j the surgery group $L_j(\Gamma)$ is isomorphic to the sum of cohomology groups*

$$\bigoplus_{i=0}^3 K_{\mathbf{F}/\text{TOP}}^i(\overline{M}, \partial \overline{M}; L_{k+i}(G))$$

where $j \equiv k + \dim M \pmod{4}$.

COROLLARY 3. For all integers j , $L_j(\Gamma) \otimes \mathbb{Q}$ is isomorphic to the direct sum

$$\bigoplus_{i=-\infty}^{\infty} H^{i+m}(\overline{M}, \partial\overline{M}; L_{i+j}(G) \otimes \mathbb{Q})$$

where $m = \dim M$, provided $\text{Wh}(G \times C^n) = 0$ for all $n \geq 0$. In this formula, $H^s(\overline{M}, \partial\overline{M}; L_t(G) \otimes \mathbb{Q})$ denotes the s -th cohomology group of the pair $(\overline{M}, \partial\overline{M})$ with (untwisted) coefficients in $L_t(G) \otimes \mathbb{Q}$.

Note that Theorem 2 and Corollary 2 are the special cases of Theorem 3 and Corollary 3 where $G = 1$.

Next we consider the spaces $\text{Diff}(\overline{M})$ and $H(\overline{M})$ of self-diffeomorphisms and self-homeomorphisms of the smooth manifold \overline{M} . The following theorem is a consequence of Theorem 2 above, Theorem 4.5 in [4] (which depends on Waldhausen’s foundational work in [20]), the stability results in [2, 15], and of the calculations of the stable pseudo-isotopy spaces given in [9].

THEOREM 4. Suppose M is orientable and compact (i.e., $\partial\overline{M}$ is empty). Then, for all integers i such that $1 \leq i < (m - 10)/7$, where $m = \dim M$, the following is true.

$$\pi_i \text{Diff}(M) \otimes \mathbb{Q} = \begin{cases} 0 & \text{if } m \text{ is even,} \\ \bigoplus_{j=1}^{\infty} H_{i+1-4j}(M, \mathbb{Q}) & \text{if } m \text{ is odd.} \end{cases}$$

More precise statements can be made about the homotopy groups of $\text{Diff}(\overline{M})$ and $H(\overline{M})$; i.e., statements about $\pi_i(\)$ instead of only about $\pi_i(\) \otimes \mathbb{Q}$ as in Theorem 4. This is done in the next theorem for $H(\overline{M})$ and uses [14]. Similar computations for the homotopy groups of $\text{Diff}(\overline{M})$ can be found in [12].

Let g_1, g_2, g_3, \dots denote the sequence of all closed geodesics in M . (Two geodesic curves are considered equal if they have the same image in M .) To each closed geodesic g_i associate a manifold S_i as follows: $S_i = I^{m-1} \times S^1$ if $w_1(g_i) = 0$, where $m = \dim(M)$ and S^1 is the circle; S_i is the total space of the unorientable I^{m-1} bundle over S^1 if $w_1(g_i) \neq 0$. Let B_1, B_2, \dots, B_n denote the connected components of $\partial\overline{M}$. For any manifold N , let $H(N, \partial N)$ denote the self-homeomorphisms of N which are the identity on ∂N , and let $\hat{H}(\overline{M})$ denote the space of all $h \in H(\overline{M})$ which are homotopic (via a homotopy $H: (\overline{M}, \partial\overline{M}) \times I \rightarrow (\overline{M}, \partial\overline{M})$) to the identity self-homeomorphism of \overline{M} . Define $\bigoplus_i H(S_i, \partial S_i)$ to be the direct limit as $j \rightarrow \infty$ of the finite cartesian products

$$H(S_1, \partial S_1) \times H(S_2, \partial S_2) \times \dots \times H(S_j, \partial S_j).$$

Define the outer automorphisms group of the pair $(\pi_1 \overline{M}, \pi_1 \partial\overline{M})$ as follows. Choose base points b_1, b_2, \dots, b_n for each of the boundary components B_1, \dots, B_n of $\partial\overline{M}$ and a base point b for M . By choosing paths which connect the b_i to b , we establish an inclusion of $\pi_1(B_i, b_i)$ into $\pi_1(\overline{M}, b)$ for each $i = 1, 2, \dots, n$. The group of outer automorphisms of the pair $(\pi_1 \overline{M}, \pi_1 \partial\overline{M})$ is the quotient group $\text{Aut}/\text{In Aut}$. Here Aut consists of all automorphisms f of $\pi_1(\overline{M}, b)$ such that (for each $i = 1, 2, \dots, n$) there is an isomorphism

$$f_i: \pi_1(B_i, b_i) \rightarrow \pi_1(B_{i'}, b_{i'}),$$

where $i \rightarrow i'$ is a permutation of the boundary components, and an element g_i in $\pi_1(\overline{M}, b)$ satisfying $f(x) = g_i f_i(x) g_i^{-1}$ for all x in $\pi_1(B_i, b_i)$. The normal subgroup InAut of Aut is the ordinary inner automorphism group of $\pi_1(\overline{M}, b)$.

THEOREM 5. (a) *There is a bijection between the set of homotopy classes of self-homeomorphisms of \overline{M} and the outer automorphism group of $(\pi_1 \overline{M}, \pi_1 \partial \overline{M})$. (Here homotopy classes refer to homotopies of maps of the pair $(\overline{M}, \partial \overline{M})$.)*

(b) *There is a continuous map*

$$f: H(B_1 \times I, B_1 \times 0) \times \cdots \times H(B_n \times I, B_n \times 0) \times \left(\bigoplus_i H(S_i, \partial S_i) \right) \rightarrow \hat{H}(\overline{M})$$

which is induced by the inclusions $g_i \subset M$ and $B_j \subset \overline{M}$, where $i > 0$ and $j = 1, 2, \dots, n$. The map f induces an isomorphism on the homotopy groups π_i , provided $i < \phi(\dim M)$, where $\phi: \mathbf{R} \rightarrow \mathbf{R}$ is a function (independent of M) such that $\lim_{m \rightarrow +\infty} \phi(m) = +\infty$.

COROLLARY 5. *The vanishing statement*

$$\pi_i H(\overline{M}) \otimes \mathbf{Q} = 0$$

is true provided $1 \leq i < \phi(m)$, where m denotes the dimension of M .

ADDED IN PROOF. Because of a result of Goodwillie, the function $\phi(m)$ satisfies the inequality $\phi(m) \geq (m - 10)/7$; this lower bound and the “stable range” in Theorem 4 can be greatly improved using Igusa’s recent work [22].

REMARK. There are analogues to Theorems 1,2,3 and 5(b) when M is not assumed to have finite volume. These analogues require the following changes. All homotopy equivalences of Theorem 1 must now have compact support; i.e., must be an embedding away from a large enough compact set. The set $[I^k \times \overline{M} \text{ rel } \partial, G/\text{Top}]$ of Theorem 2 must be replaced by the set of compactly supported homotopy equivalence classes of compactly supported maps. The cohomology groups of Theorem 3 must be replaced by cohomology with compact support. The spaces $H(\)$ and $\hat{H}(\)$ of Theorem 5(b) must be replaced by the spaces of compactly supported self-homeomorphisms and the homotopy H must have compact support.

Theorems 1–5 are proved in [12]. In sequels to this paper we will pursue analogues of Theorem 1–5 for wider classes of manifolds, including nonpositively curved locally symmetric spaces of higher rank.

We give now an indication of the proof for the special case of Theorem 1 given in Corollary 1. Our proof follows the same general outline as the arguments given in papers [8 and 9], in which the lower algebraic K -groups of the fundamental group of any real hyperbolic manifold were computed together with a determination of the weak homotopy type of its stable pseudo-isotopy space. There are two major ingredients:

- (1) a foliated control theorem for structure spaces,
- (2) the dynamics of the geodesic flow (cf. [1]) together with other geometric ideas.

The foliated control theorem for structure spaces, used in the proof of Theorem 1, is proved in [11]. It extends the unfoliated control results of [3 and 19]. Unlike the arguments in papers [8, 9 and 10] where we needed only foliated control theorems for foliations having one-dimensional leaves, the proof of Theorem 1 requires a structure space foliated control theorem for foliations having higher-dimensional leaves.

So as to abbreviate the many complications which arise in the proof, we assume that M is an odd-dimensional manifold. The geometry of M enters into the proof of Theorem 1 in the following way. It enables us to construct a compact stratified space F with two strata B and T where B is the bottom stratum and T is the top stratum. The fundamental group Γ of M acts on F preserving these strata. Let $F(M)$ denote the orbit space of $\tilde{M} \times F$ under the diagonal action of Γ where \tilde{M} is the universal cover of M ; i.e.,

$$F(M) = \tilde{M} \times_{\Gamma} F.$$

Note $F(M)$ is the total space of a fiber bundle over M with fiber F . Thus it is a stratified space having two strata \mathbf{B} and \mathbf{T} where

$$\mathbf{B} = \tilde{M} \times_{\Gamma} B \quad \text{and} \quad \mathbf{T} = \tilde{M} \times_{\Gamma} T.$$

Furthermore $F(M)$ is equipped with a foliation \mathcal{F} which restricts to smooth foliations \mathcal{F}_B and \mathcal{F}_T for the strata \mathbf{B} and \mathbf{T} , respectively. The geometry of M gives rise to additional structures, called markings, on the leaves of \mathcal{F}_B and \mathcal{F}_T . Each leaf of \mathcal{F}_T is marked with a geodesic while each leaf of \mathcal{F}_B is marked with an asymptotic vector field. These markings vary smoothly from leaf to leaf within each stratum.

Consider a homotopy equivalence $h: N \rightarrow \bar{M}$ from a compact manifold N which restricts to a homeomorphism near $h: \partial N \rightarrow \partial \bar{M}$. (See the comment following the statement of Corollary 1.) Thus $h: N - \partial N \rightarrow M$ is a homotopy equivalence with compact support. Let F^* denote the total space of the pullback of the bundle $F(M) \rightarrow M$ along h restricted to $N - \partial N$, and let $h^*: F^* \rightarrow F(M)$ denote the natural lifting of $h: N - \partial N \rightarrow M$ to F^* . We claim there is a compactly supported homotopy h_t^* , $t \in [0, 1]$, with $h_0^* = h^*$ and such that h_1^* is a homeomorphism over \mathbf{B} and over the complement of a tubular neighborhood for the stratum \mathbf{B} in $F(M)$, and such that h_1^* is "blocked up" over this tubular neighborhood. To get such a homotopy, we first "flow" h^* over the bottom stratum in the direction of the asymptotic vector fields of the leaves of \mathcal{F}_B , thereby gaining one-dimensional foliated control for h^* over the bottom stratum \mathbf{B} . Now apply the structure space foliated control theorem from [10 and 11] (for the one-dimensional foliation of \mathbf{B} consisting of the flow lines) to construct the homotopy h_t^* over \mathbf{B} . To get h_t^* over the top stratum (away from a tubular neighborhood of \mathbf{B}), we use the geodesic markings on the leaves of \mathcal{F}_T together with the structure space foliated control theorem for higher dimensional leaves of special geometric type, proved in [11]. Finally, h_t^* is extended over the tubular neighborhood of \mathbf{B} by the fibered version of the foliated control theorem for foliations with one-dimensional leaves, also proved in [11].

Surgery procedures now allow us to “desuspend” the homotopy h_t^* of the preceding paragraph to get a homotopy $h_t: N \rightarrow \overline{M}$, $t \in [0, 1]$, from $h = h_0$ to a homeomorphism h_1 and such that h_t restricted to ∂N is the constant homotopy. It is at this step that we use the extra assumption: M has odd dimension, which implies F has index one.

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