BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 18, Number 2, April 1988
©1988 American Mathematical Society
0273-0979/88 \$1.00 + \$.25 per page

Buchsbaum rings and applications: An interaction between algebra, geometry and topology, by Jürgen Stückrad and Wolfgang Vogel. Springer-Verlag, Berlin, Heidelberg, New York, London, Paris and Tokyo, 1986, 286 pp., \$65.50. ISBN 3-540-16844-3

What are Buchsbaum rings and why should one care? These are the two obvious questions when confronted by this recent book. The answer to the first question is quick enough but a bit technical. The answer to the second is, naturally enough, the contents of the book. In order to put the concept of Buchsbaum rings into its proper context, the authors begin with a discussion of multiplicity. One of the oldest theorems in algebra is that a polynomial of degree n has n roots if you count the number correctly. This theorem becomes much more complicated when generalized to more complicated rings than just the ring of polynomials in one variable over a field. Indeed one of the recent triumphs of mathematics has been the development of modern intersection theory, which has enabled the naive idea of multiplicity to have a firm foundation. A first approximation to the multiplicity of a collection of polynomials at a point is to consider the length of the resulting quotient ring. If we start with a field K and a polynomial $f(x)$, then the $\dim_K K[x]/(f(x))$ is precisely the degree of $f(x)$. In the more general case of polynomials in several variables one can consider

$$\dim_K K[x_1,\dots,x_n]/(f_1(x_1,\dots,x_n),\dots,f_m(x_1,\dots,x_n))$$

as a measure of the multiplicity of the polynomials if the dimension is finite. However, if one considers arbitrary noetherian rings R rather than fields, the answer does not behave correctly without some technical assumptions on the ring considered. If the ring in question is Cohen-Macaulay (that is, the dimension of R is the codim of R) and one takes a set of $\dim R$ polynomials which generate an ideal J primary to the maximal ideal, then the answer is correct. That is, the length of R/J (the analogue of $\dim_K K[x]/(f(x))$) is the multiplicity of the set of polynomials. Buchsbaum took this point of view in his C.I.M.E. lectures in 1965. He wrote, "It would of course be hoped that the difference between length and multiplicity could be determined by the difference $\dim R - \operatorname{codim} R$ and/or other invariants yet to be found." Thus were Buchsbaum rings created. One calls a local ring a *Buchsbaum ring* if, for all ideals J that are generated by $\dim R$ elements and are primary to the maximal ideal, one has that length (R/J) minus multiplicity (R/J) is a constant not depending on J. Cohen-Macaulay rings are those Buchsbaum rings for which the constant is zero.

This portion of commutative ring theory is one in which all the problems occur for local rings. Local rings are noetherian rings with only one maximal ideal. A ring is Buchsbaum if all of its localizations are Buchsbaum rings. Similarly, a ring is Cohen-Macaulay if all its localizations are Cohen-Macaulay rings. Many problems in commutative ring theory behave this way. The question of whether a given projective module is free is a typical example of a question that cannot be reduced to the local case since all projectives over a local ring are free.

This answers the question of what is a Buchsbaum ring. The second question is more difficult. Of course, Buchsbaum rings constitute a serious but technical part of mathematics and hence have some intrinsic interest. One can look for more. Another way of placing this family of rings in its context is to go back about thirty years to the dawn of homological algebra. At that time there were two significant problems about local rings, which the standard methods of the time could not solve. The most important local rings then (and now) were the regular local rings. These are the rings that correspond to the smooth points on an algebraic variety. They are the local rings such that their maximal ideal can be generated by d elements, where d is the dimension of R. The two important problems were whether regular local rings were always unique factorization domain and whether localizations of regular local rings were again regular local. Auslander-Buchsbaum and Serre used the new homological algebra to solve these problems in the affirmative. These papers established homological algebra as a significant tool in commutative ring theory. In many cases the invariant one wants to understand is the dimension of quotient rings. Homologically the invariant that is like dimension and that behaves the best is the codimension (now often called depth). Naturally enough, the class of rings for which these two invariants are the same is important. They turn out to be the Cohen-Macaulay rings. The interplay between dimension and depth and the corresponding focus on Cohen-Macaulay rings and modules has been a major theme in commutative ring theory. Much of the recent work of Hochster, Peskine, Szpiro, Roberts, and others has this question as a focus.

One way of viewing Buchsbaum rings is that they are just slightly more general than Cohen-Macaulay rings. A more modern way of defining Cohen-Macaulay rings is to say that they are the local rings for which the only nonzero local cohomology group is at the dimension of R. For a Buchsbaum ring all the other local cohomology groups are annihilated by the maximal ideal. The converse is nearly true. The authors call a ring quasi-Buchsbaum if the local cohomology groups except at the dimension of the ring are annihilated by the maximal ideal. The extra assumption needed to make the ring be Buchsbaum is rather technical. If one feels that Cohen-Macaulay rings are important, then Buchsbaum rings which just miss being Cohen-Macaulay gain stature by association. Indeed, as the authors point out, several of the famous examples of rings which fail to be Cohen-Macaulay are in fact Buchsbaum.

Still, the question remains, why are these rings worth an entire book? The answer lies in the book itself. What the book gives us is a very general treatment of several areas of commutative ring theory and related areas of geometry and topology, with a discussion of the consequences of those areas for the class of Buchsbaum rings. Thus, one can read the book as a general commutative ring theory book and skip those parts that are particular to Buchsbaum rings much in the way that some people read *Moby Dick* and skip the parts about whaling. Of course you may get interested in the whales in the process.

There is a wealth of general information in the book. After a lovely Chapter 0 in which the basic ideas of commutative algebra are reviewed and a rather technical Chapter 1, which characterizes Buchsbaum rings and modules, one is treated to a series of topics of more general interest. The first of these is a solid treatment of the Hochster-Reisner-Stanley theory of ideals in polynomial rings generated by monomials. Briefly, if one has a finite simplicial complex with vertices $x_0, \ldots, x_n$, then one can form an ideal generated by the monomials that are products of distinct x's so that the complex formed by those x's is not a member of the complex. There is a fascinating interplay between the topology of the simplicial complex and the algebra of $K[x_0, \ldots, x_n]$ modulo the ideal. Perhaps the easiest result to state is that the complex represents a homology complex if and only if the resulting ring is a Buchsbaum ring. There are also criteria for when the ring is Cohen-Macaulay. These criteria are older than the criteria for being Buchsbaum (they go back to the thesis of Reisner). However, the Cohen-Macaulay conditions are more technical than the Buchsbaum conditions. This fact gives one a feeling that the Buchsbaum property is more natural than one might guess at first glance. Stanley has used this series of ideas to prove theorems in combinatorics. The authors discuss this and find Buchsbaum rings there also. There is also a treatment of ideals generated by general (rather than square-free) monomials by the simple but expedient method of adding extra variables to make new names for old variables.

The other chapters proceed along similar lines. Topics such as liaison, Rees rings, and symmetric algebras are discussed, first from the general viewpoint and then for the specific case of Buchsbaum rings. A real strength of the book is the abundance of concrete examples to test the theory on. Another strength is the large set of references. The book has some curious weaknesses. I was fascinated by the authors occasional lapses into their native German. "Exact" and "exakt" occur just a few words apart. "Auf" is used in the middle of a sentence for "of". The printing and paper quality is not what one would expect from Springer until one realizes that they are only the distributors of the book for nonsocialist countries. This is not likely to be a book that the average mathematician would want on his shelf. However the breadth of subjects that are brought to bear on this very technical subject is impressive. One can leaf through it and find topics of interest or turn to it for a nice treatment of many areas of the subject.

BIBLIOGRAPHY

David Buchsbaum, *Complexes in local ring theory*, Some Aspects of Ring Theory, C.I.M.E., Roma, 1965, pp. 223–228.

W. Fulton, *Intersection theory*, Springer-Verlag, New York, Heidelberg, Berlin, Tokyo, 1984.

M. Hochster, *Topics in the homological theory of modules over commutative rings*, CBMS Regional Conf. Ser. in Math, no. 24, Amer. Math. Soc., Providence, R. I., 1975.

J.-P. Serre, *Algèbre locale—multiplicités*, Lecture Notes in Math., vol. 11, Springer-Verlag, Berlin, Heidelberg, New York, 1965.

E. GRAHAM EVANS, JR.