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Introduction to complex hyperbolic spaces, by Serge Lang. Springer-Verlag, New York, Berlin, Heidelberg, 1987, viii + 271 pp., $58.00. ISBN 0-387-96447-9

There is a (possibly apocryphal) story about Émile Picard that I heard as a graduate student. Just returned from attending a conference, Picard excitedly informed his father-in-law, Hermite, of a beautiful problem suggested by some work of Weierstrass that he had heard about there: was it possible for a nonconstant entire function on $\mathbf{C}$ to omit two values? Hermite in turn showed Picard some theorems he had proved while Picard was away involving modular functions, including the λ-invariant. The story is a useful cautionary tale for young mathematicians—go to a conference and you may pick up a good problem; stay away and you might prove a nice theorem. Picard, this one time, did both: the λ-invariant gives a holomorphic covering map from the upper half-plane $\mathcal{H}$ or, equivalently, the unit disc Δ, to $\mathbf{C} - \{0,1\}$. Any holomorphic map $f\colon \mathbf{C} \to \mathbf{C} - \{0,1\}$ lifts to a holomorphic map $\tilde{f}\colon \mathbf{C} \to \Delta$. By Liouville's Theorem, such a map must be constant, and therefore so is f. The complement of any two distinct points of $\mathbf{C}$ is analytically equivalent to the complement of any other two. Thus Picard showed that no nonconstant entire function on $\mathbf{C}$ can omit two values.

One can think of Picard's Theorem as stating that there are no nonconstant holomorphic maps $\mathbf{C} \to \mathbf{C}P^1 = S^2$ which omit three points. Picard went on to consider the case of nonconstant holomorphic maps $\mathbf{C} \to X = \overline{X} - S$, where $\overline{X}$ is a compact Riemann surface and S is a finite set of points. By the Uniformization Theorem, the universal cover $\widetilde{X}$ of X is either $\mathbf{C}P^1$, $\mathbf{C}$, or Δ. An elementary analysis of what the covering transformations can be revealed

$$\widetilde{X} = \mathbf{C}P^1 \to X = \mathbf{C}P^1,$$
$$\widetilde{X} = \mathbf{C} \to X = \mathbf{C},\ \mathbf{C}^*,\ \text{or } \mathbf{C}/\Lambda \text{ for some lattice } \Lambda.$$

In all other cases, $\widetilde{X} = \Delta$ and the same argument via Liouville's Theorem shows that there are no nonconstant holomorphic maps $f\colon \mathbf{C} \to X$. A convenient shorthand for this result is: For $X = \overline{X} - S$ of dimension one,

$$\text{Picard's Theorem holds for } X \Leftrightarrow \text{The Euler number } \chi(X) < 0.$$

Given X above with $\chi(X) < 0$, there are of course lots of nonconstant holomorphic maps $f\colon \Delta \to X$. Schottky and Landau noticed that the elementary Schwartz Lemma, which says that a holomorphic map $f\colon \Delta \to \Delta$ with $f(0) = 0$ has $|f'(0)| \le 1$, translates into the following quantitative result: given $p \in X$ and a tangent vector $v \in T_p(X)$, there exists a constant R such that there is no holomorphic map f from the disc $\Delta(r)$ with $r \ge R$ with $f(0) = p$, $f'(0) = v$. It is now natural to define a length function

$$|v|_X = \inf\{1/r \mid \text{There exists a holomorphic map } f\colon \Delta(r) \to X \text{ with } f(0) = p, f'(0) = v\}.$$

This length function is in this case a Hermitian metric; indeed, it is the pushdown via the universal cover $\Delta \to X$ of the *Poincaré metric* of constant negative curvature on Δ.

With this in mind, given any complex manifold (or even complex space) X, we can define a length function $|v|_X$ on tangent vectors $v \in T_p(X)$. This need no longer be a Hermitian metric. It may be zero for $v \neq 0$; it may behave nonlinearly as v varies in $T_p(X)$ for p fixed, and it may not be continuous in p. It is nevertheless an object which is both canonical and highly useful. It is called the *Royden pseudo-length function.* Given p, $q \in X$ and a smooth $\mathcal{C}^1$ curve $\gamma\colon [0,1] \to X$ between them, the integral

$$L_X(\gamma) = \int_0^1 |\gamma'(t)|_X \, dt$$

turns out to make sense because $|\ |_X$ is upper semicontinuous. We can then define the *Kobayashi pseudo-distance function*

$$d_X(p,q) = \inf\{L_X(\gamma) \mid \gamma \text{ is a } \mathcal{C}^1 \text{ path from } p \text{ to } q\}.$$

It is always ≥ 0, symmetric, and satisfies the triangle inequality. It need not be a metric (i.e. $d_X(p,q) > 0$ if $p \neq q$ may fail). For example, $d_{\mathbf{C}}(p,q) = 0$ for all p, $q \in \mathbf{C}$. Kobayashi, who defined d_X differently, defined X to be *hyperbolic* if d_X is indeed a metric.

One elementary property of d_X is that every holomorphic map $f\colon X \to Y$ of complex spaces is distance decreasing for the Kobayashi pseudo-distance, i.e. $d_Y(f(p), f(q)) \leq d_X(p,q)$. By an elementary computation, d_Δ is the Poincaré metric. This said, the results of Picard, Schottky, and Landau mentioned thus far may be recapitulated by saying that for a punctured Riemann surface $X = \overline{X} - S$

$$X \text{ is hyperbolic} \Leftrightarrow \chi(X) < 0.$$

If one is interested in knowing which algebraic varieties are hyperbolic, this gives a complete answer in dimension one.

As soon as we consider higher-dimensional varieties, the Uniformization Theorem ceases to be available. Through the work of F. Nevanlinna and Ahlfors came the realization that in the curve case, what is essential about the condition $\chi(X) < 0$ is that X possesses a Riemannian metric of curvature ≤ -1. The key step is the Ahlfors Lemma:

If a Riemann surface X possesses a Riemannian metric $\|\ \|$ with Gaussian curvature $\leq -k < 0$, then for any holomorphic map $f\colon \Delta \to X$, $k\|df(v)\| \leq |v|_\Delta$ for all $z \in \Delta$ and $v \in T_z(\Delta)$.

An immediate corollary is that $|\ |_X \geq k\|\ \|$.

The Ahlfors Lemma is remarkably susceptible of generalization. For higher dimensions, the theorem is true for Hermitian metrics if Gaussian curvature is replaced by holomorphic sectional curvature. Further, one can get by with a length function rather than a Hermitian metric, and this turns out to be important for applications. The main problem then becomes how to construct metrics of negative curvature on spaces that one suspects of being hyperbolic. Here the essential insight, although it is hinted at classically, appears in the work of Griffiths. He realized that it is possible to construct metrics of negative curvature by using algebro-geometric properties of the space. The simplest example is for X a nonhyperelliptic compact Riemann surface of genus ≥ 2. If $\omega_1, \omega_2, \ldots, \omega_g$ is a basis for the holomorphic one-forms of X, then one takes

$$\|v\|^2 = |\omega_1(v)|^2 + \cdots + |\omega_g(v)|^2.$$

This is the Bergmann metric, and it has strictly negative Gaussian curvature. In the hyperelliptic case, the curvature of this metric vanishes at the Weierstrass points. For that case, if $\phi_1, \ldots, \phi_{5g-5}$ is a basis for the cubic differentials, i.e. for $H^0(X, (\Omega_X^1)^{\otimes 3})$, then

$$\|v\|^6 = |\phi_1(v)|^2 + \cdots + |\phi_{5g-5}(v)|^2$$

gives a metric of strictly negative curvature. It is possible to make similar constructions in higher dimensions using holomorphic 1-forms, symmetric differentials, and jet forms.

One truly unanticipated development in the subject was R. Brody's remarkable thesis. Using a simple but extraordinary trick from complex analysis (*Brody's Reparametrization Lemma*), Brody proved: For a compact complex manifold X,

X is hyperbolic $\Leftrightarrow$ X admits no nonconstant holomorphic maps from $\mathbf{C}$.

Indeed, for a nonhyperbolic X, he produces for any Hermitian metric H on X a nonconstant holomorphic map $f: \mathbf{C} \to X$ which is distance decreasing, letting $\mathbf{C}$ have the Euclidean metric. As a sample of the strength of this result, we note that if X is a compact submanifold of a complex torus T and X contains no subtori of T, and taking H to be the restriction of the Euclidean metric from T to X, then there are no nonconstant distance decreasing holomorphic maps $\mathbf{C} \to X$, and thus such an X is hyperbolic. There are variants of Brody's method for attacking noncompact complex spaces by which a number of interesting questions have been settled.

There are two reasons why the appearance of a new book in this area—hyperbolic spaces and their relationship to algebraic geometry—is especially natural at this time. The first is that the subject, under the impetus of the work of Chern, Grauert, and Griffiths, enjoyed a period of progress and expansion in the late 60s and throughout the 70s. The 80s, in comparison, have been quiet. Kobayashi's book [**1**]—the first on hyperbolic manifolds—appeared in 1970, close to the beginning of this active period. His book antedates Brody's work and most of the theorems which so richly justify Kobayashi's definition of hyperbolic spaces. By the end of the 70s, the essential tools of the subject lay scattered throughout the literature. Lang's book, together with his survey article [**2**], assembles these in one place for the first time and elucidates the connections between different individuals' approaches.

The second reason is that the spectacular successes of Faltings and the intriguing work of Vojta have elicited an interest in hyperbolic spaces on the part of number theorists. The easiest way to see why this should be so is to ask what hypothesis on a complete variety X should replace in higher dimensions the hypothesis $g \geq 2$ of Mordell's Conjecture for curves. A natural guess is that one should take X hyperbolic. This, however, only scratches the surface of the interrelationship betwen the two subjects; a fuller discussion is available in Lang's survey article or in Vojta's book [**3**].

There is a great deal more covered in Lang's book than I have discussed here, e.g., Nevanlinna theory and the defect relations of Carlson-Griffiths, surely the foremost success of differential-geometric methods in the subject. Indeed, there is little of importance in the area that Lang set out to cover which he has not managed to include in either his book or survey article. If I had a student in this area, I would surely point to these two sources and say, "This is what you ought to learn." There are, as one might expect in a work of this scope, better places to learn some of the topics covered; however, nowhere else are even half of these topics all to be found together. Chapters 3 and 7, for example, contain material available nowhere else in book form. Lang has performed a tremendously important service to the subject.

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BULLETIN (New Series) OF THE
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Spectral geometry: Direct and inverse problems, by Pierre H. Bérard, with an appendix by G. Besson. Lecture Notes in Mathematics, Vol. 1207, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1986, xiii + 272 pp., $23.40. ISBN 3-540-16788-9

This interesting book deals with certain direct problems in Riemannian geometry. The notions of direct and inverse problems are not exact, but in general, by a direct problem one means a problem in which the goal is to derive information about the spectrum of the Laplace operator on a Riemannian manifold M from other geometric data associated with the manifold, while an inverse problem is one in which the deduction goes the other way, i.e., information about the spectrum is used to derive different geometric information about M.

For example, the Faber-Krahn result that among smooth domains in R^n of fixed volume the first Dirichlet eigenvalue is minimized by the ball is of a direct character, whereas the fact that the dimension of such a domain is determined by its Dirichlet spectrum is of an inverse character.

There is a very large body of results on these topics, with certain persistent and central themes, e.g., the effort to understand the asymptotic behavior of the spectrum. Strikingly, this behavior only depends, up to an asymptotically correct first approximation, on the dimension and Riemannian volume of M (cf. [**M-P**]). A related development has been the discovery of relationships which, formally at least, convey the exact geometric content of the spectrum,