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BAND-LIMITED FUNCTIONS: L^p-CONVERGENCE

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I. Introduction and statement of results. In this paper we shall be concerned with a set of functions, namely the prolate spheroidal wave functions, which have been considered by several authors especially in relation with problems of communication theory. These functions appear naturally in the formulation of the uncertainty principle: a very interesting problem there is that of making signals $f = f(t)$ of total power $\|f\|_2 = 1$ with both

$$\tau^2 = \int_{-T}^{+T} |f(t)|^2\,dt \qquad \omega^2 = \int_{-\Omega}^{+\Omega} |\hat{f}(\xi)|^2\,d\xi$$

as close to 1 as possible for given positive numbers T and Ω, where

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} e^{-2\pi i x\cdot\xi} f(x)\,dx$$

denotes the Fourier transform.

$\tau = 1$ means that the signal is confined to the period $|t| \leq T$ or that f is time-limited.

$\omega = 1$ means that its power spectrum is confined to the band $|\xi| \leq \Omega$ or that f is band-limited.

It is a well-known fact about the Fourier transform that both τ and ω cannot be equal to 1 at the same time. In a series of papers by Landau, Pollak, and Slepian [**4, 5, 7, 8**] the following result was proved: the pairs (τ,ω) corresponding to signal f with $\|f\|_2 = 1$ describe the region defined by the inequalities

$$\cos^{-1}(\tau) + \cos^{-1}(\omega) \geq \cos^{-1}\sqrt{\lambda_0}, \qquad 0 \leq \tau \leq 1, 0 \leq \omega \leq 1$$

(with the proviso that if τ or $\omega = 1$ ($= 0$), then the other is > 0 (< 1)) where

$$\lambda_0 = \sup\left\{\int_{-T}^{+T} |f(t)|^2\,dt \middle| f \in L^2(\mathbf{R}), \|f\|_2 = 1, \operatorname{supp}(\hat{f}) \subset [-\Omega, +\Omega]\right\}.$$

Another interesting result is the so-called dimension theorem. Roughly speaking it says that the dimension of the set of signals band-limited to the interval $[-\Omega, +\Omega]$ and "concentrated" in $[-T, +T]$ is $4T\Omega$. We refer to the above-mentioned papers for a more precise statement.

In the proof of these results, the operator

$$Tf(x) = \int_{-T}^{+T} \frac{\sin 2\pi\Omega(t-s)}{\pi(t-s)} f(s)\,ds$$

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plays a central rôle. It is a convolution kernel whose Fourier transform is an indicator function. The operator is positive and by the standard theory it has a countable family of eigenvalues $1 > \lambda_0 > \lambda_1 > \lambda_2 > \cdots$ decreasing to 0.

The associated family of normalized eigenfunctions, $\{b_n\}$, can be chosen to be real-valued and they form an orthonormal basis of the class of band-limited functions:

$$B_2 = B_2(\Omega) = \{f \in L^2(\mathbf{R}) \mid \operatorname{supp}(\hat{f}) \subset [-\Omega, +\Omega]\}.$$

Furthermore the family $\{\lambda_n^{-1/2} b_n(t)\chi_{[-T,+T]}(t)\}$ is also an orthonormal basis of the time-limited functions, i.e., of the space $L^2[-T, +T]$.

In this work we extend the above results to the case of L^p, $p \neq 2$. To do that we define

$$B_p = B_p(\Omega) = \{f \in L^p(\mathbf{R}) \mid \operatorname{supp}(\hat{f}) \subset [-\Omega, +\Omega]\}, \qquad 1 \leq p \leq \infty.$$

Then B_p is a Banach space with the norm induced by $L^p(\mathbf{R})$ and we have the continuous inclusion $B_p \subsetneq B_{p'}$ if $p < p'$. Furthermore the family $\{b_n\}$ is contained in B_p for every $p > 1$. We can now formulate our main result:

THEOREM. *The system $\{b_n\}$ is a basis of the space $B_p(\Omega)$ if and only if* $4/3 < p < 4$.

II. A certain amount of preparation. It is easy to see that the functions b_n depend only upon the product $c = 2\pi\Omega T$. Therefore we may, without loss of generality, normalize them by taking $T = 1$. In that case, it is a well-known fact that they coincide on the interval $[-1, +1]$ with the eigenfunctions of the Sturm-Liouville problem:

$$(1 - x^2)y''(x) - 2xy'(x) + (\chi - c^2x^2)y(x) = 0$$

with suitable boundary conditions. This equation arises on separating the 3-dimensional scalar wave equation in a prolate spheroidal coordinate system.

One obtains then the expansion

$$(*) \qquad b_n(x) = 2\sqrt{\Omega}\, u_n^{-1} \sum_{k=q(n)}^{\infty} (-1)^k d(k,n) \sqrt{\frac{2n+4k+1}{2}}\, j_{n+2k}(2\pi\Omega x).$$

Here $q(n) = [-(n-1)/2]$, j_n denotes the spherical Bessel function of order n, and $u_n^2 = \sum_{k=q(n)}^{\infty} |d(k,n)|^2$, and the $d(k,n)$ satisfy suitable estimates.

It is now clear that, from the point of view of the proof of our theorem, the particular value of Ω is irrelevant. Consequently, to simplify the notation, we shall take $2\pi\Omega = 1$.

III. Sketch of the proof. The main point of the proof is to show the uniform boundedness in B_p, $4/3 < p < 4$, of the partial summation operators

$$S_N f(x) = \int_{-\infty}^{+\infty} K_N(x,y) f(y)\, dy, \qquad \text{where } K_N(x,y) = \sum_{n=0}^{N} b_n(x) b_n(y).$$

We use the series $(*)$ to decompose K_N into several parts. Essentially, the problem is reduced to the study of the kernel

$$K_N(x,y) = \sum_{n=0}^{N} (2n+1) j_n(x) j_n(y).$$

(This study also gives us the analogous theorem for the spherical Bessel functions.)

The identity $j_n(x) - j_{n+2}(x) = \sqrt{2\pi/x}\,J'_{n+3/2}(x)$, where J_k denotes the Bessel function of order k, allows us to obtain

$$K_N(x,y) = Q_N(x,y) + Q_N(y,x) + R_N(x,y),$$

where

$$Q_N(x,y) = \frac{x^{1/2} y^{1/2}}{x-y} J'_{N+1/2}(x) J_{N+1/2}(y).$$

The analysis of the term $R_N(x,y)$ follows a more standard path. The operator associated to $Q_N(x,y)$ is

$$T_N f(x) = x^{1/2} J'_{N+1/2}(x) H(y^{1/2} J_{N+1/2}(y) f(y))(x),$$

where H denotes the Hilbert transform.

Therefore our problem becomes a two-weight inequality for the Hilbert-transform. To get the needed uniform estimates, no one of the known results for the operator H seems to apply directly, and we are forced to produce a rather careful analysis, using the stationary phase method mainly, of the size of both $J_{N+1/2}(x)$ and $J'_{N+1/2}(x)$ in the delicate region, i.e. when the argument x is of the order of magnitude of the index $N+\frac{1}{2}$.

We prove that for every p, $4/3 < p < 4$, there exists a finite constant C_p such that

(i) $\|V_1 H(V_2 f)\|_p \le C_p \|f\|_p$,
(ii) $\|V_1 M(V_2 f)\|_p \le C_p \|f\|_p$.

Here $V_1(x) = |x^{1/2} J'_{N+1/2}(x)|$, $V_2(x) = |x^{1/2} J_{N+1/2}(x)|$ and M denotes the Hardy-Littlewood maximal function.

The analogous problem for the time-limited functions can be reduced to familiar properties of the Legendre polynomials. It is known that these polynomials form a basis of $L^p[-1,+1]$ if and only if $4/3 < p < 4$.

We have also extended several results of the classical harmonic analysis of Fourier series to the case of these expansions on prolate spheroidal wave functions. Details will appear elsewhere.

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