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CONTROL VARIATIONS WITH AN INCREASING NUMBER OF SWITCHINGS

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1. Introduction. The purpose of this paper is to introduce new families of control variations and exhibit how they lead to high-order conditions for controllability which cannot be obtained by the usual methods. Also, we explain why the underlying phenomenon is likely to be very important for the synthesis of (time-optimal) feedback.

Suppose $X(x)$ and $Y(x)$ are real analytic vectorfields on R^n with $X(0) = 0$. They give rise to the single-input affine control system

$$(1) \qquad \begin{cases} \dot{x} = X(x) + uY(x), & |u(t)| \leq \varepsilon_0, \\ x(0) = 0, \end{cases}$$

where the control u is a measurable function defined on some interval $[0, T]$ with bound $\varepsilon_0 > 0$. The solution to (1) with control u is denoted by $x(t, u)$. The *attainable set* at time t (with control bound ε_0) is $\mathcal{A}_{\varepsilon_0}(t) = \{x(t, u) : |u(\cdot)| \leq \varepsilon_0\}$. The system (1) is *small-time locally controllable* (STLC) if $\mathcal{A}_{\varepsilon_0}(t)$ contains the rest solution $x \equiv 0$ in its interior for all ε_0, $t > 0$.

Let $L(Y, X)$ be the Lie algebra generated by the vectorfields Y and X, and $L(Y, X)(p) = \{W(p) : W \in L(Y, X)\}$ for a point $p \in R^n$. A consequence of the *Hermann-Nagano Theorem* is [**13**]: If $L(Y, X)(0)$ is the full tangent space at zero then int $\mathcal{A}_{\varepsilon_0}(t) \neq \emptyset$ for all ε_0, $t > 0$, and in the case of analytic vectorfields the converse is true, also. Sometimes referred to as the *Second Nagano Theorem* is [**10**], loosely speaking: Up to diffeomorphisms all local properties of (1) are determined by the values of the iterated Lie brackets of X and Y at zero. In view of this it is natural to look for necessary and sufficient conditions for STLC in terms of Lie brackets of Y and X at 0. In recent years substantial progress in this direction has been made, e.g. [**2, 4, 5, 8, 12**].

All sufficient conditions for STLC known today, and also the *Pontriagin Maximum Principle* and the *High Order Maximum Principle* [**6**], have in common that their proofs crucially rely on continuously parametrized families of piecewise constant control variations $\{u_s\}_{s \geq 0}$ (in this case of the zero control $u_0 \equiv 0$) with a fixed number of jumps, the parameter s being closely related to the amplitude of the control variation u_s and/or the length of the time intervals on which it is different from the reference control.

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The families of (also piecewise constant) control variations introduced here will be parametrized by a *discrete* parameter closely related to the *number of jumps* which will *grow to infinity* as s approaches zero.

To obtain sufficient conditions for controllability (or, equivalently, necessary conditions for optimality) one typically uses these families of control variations to generate approximating cones (of tangent vectors) to the attainable set(s) which lead to the desired results via a suitable *open mapping theorem* (e.g. [**3**]).

The underlying phenomenon is likely to also be very important for the study of regularity of optimal controls (and thus for the synthesis of optimal feedback): It is known that for linear systems the optimal controls may be taken to be *bang-bang* (i.e. with values in the vertices of the control set only, here ± 1) with an a priori bound on the number of switchings [**11**]. In the nonlinear case *singular arcs* may occur, but recently for low dimensional generic systems bounds for the number of bang/singular pieces of the optimal controls (trajectories) have been obtained [**1**, **7**]. Finally, optimal controls with accumulation points of switching times may occur, giving rise to *Fuller curves*. The relation between such controls with infinitely many switchings and the families of controls with an increasing number of switchings introduced here might be another interesting object of study, but one which here we shall not pursue further.

Also, the systems which only can be controlled by means of these new fast switching controls typically have attainable sets that grow at very different rates in (at least two) opposite directions, which may be of interest in the theory of PDOs since the attainable set as considered here is closely related to the region on which a strong maximum principle holds [**9**] (for the hypoelliptic operator associated to the control system (1)).

2. The result. We will use these new families of control variations to show that a certain system on R^4 is STLC; and we also prove that the use of these fast switching variations is essential, in the sense that the system cannot be controlled (in small time) by using the standard families of variations.

The given system stands for a wide class of systems of form (1) all exhibiting this behavior; but for the clarity of the argument we will do the calculations for this one typical system only. (A general theorem will be the subject of a forthcoming paper.)

The system under consideration is

$$(2) \qquad \begin{cases} \dot{x}_1 = u, & x(0) = 0, \\ \dot{x}_2 = x_1, & |u(t)| \leq \varepsilon_0, \\ \dot{x}_3 = x_1^3, & \\ \dot{x}_4 = x_3^2 - x_2^7. & \end{cases}$$

Writing this system in the standard form $\dot{x} = X + uY$, one easily computes the two brackets which in this case ultimately determine whether the system is STLC (here $(\mathrm{ad}V, W) = [V, W]$ and $(\mathrm{ad}^{i+1}V, W) = [V, (\mathrm{ad}^i V, W)]$):

$$W^1(0) = \frac{1}{72}(\mathrm{ad}^2(\mathrm{ad}^3 Y, X), X)(0) = \partial/\partial x_4$$

and

$$W^2(0) = \frac{1}{7!}(\mathrm{ad}^7[X,Y],X)(0) = \partial/\partial x_4.$$

By a standard homogeneity argument (which essentially amounts to counting the factors X and Y in the brackets W^1 and W^2), one expects for sufficiently small time t the definite term x_3^2 in the last component of (2) to dominate the indefinite term x_2^7, i.e. $x_4(t,u) \geq 0$ for t small, or more precisely one expects the intersection of the negative x_4-axis with the attainable set to be empty for small positive times and control bounds. However, here we show:

CLAIM 1. The system (2) is STLC.

CLAIM 2. If for fixed $N \in Z^+$ the class of admissible controls is restricted to those s.t. the function $t \to x_1(t,u) = \int_0^t u(s)\,ds$ changes sign at most $N-1$ times, then $x_4(T,u) \geq 0$ if $x_1(T,u) = 0$ and $N^7 \leq \varepsilon^{3/4}T^{7/2}$.

It can be shown [**5**] that the system (2) is not STLC if x_2^7 is replaced by x_2^m, $m \geq 8$.

Note, that Claim 2 in particular contains the two cases when u is piecewise constant with at most N jumps and when u is piecewise smooth and changes sign at most N times.

The consequences for the synthesis of (time-) optimal feedback are not yet completely understood: From Claim 2 we know that the optimal controls/trajectories must be bad, however the *switching surfaces* still may be nice, e.g. a locally finite union of embedded manifolds.

In the following we outline the proofs of the two claims, emphasizing the role of the new control variations.

To prove Claim 1, we show that there are constants $C, m > 0$ such that for all positive times $T > 0$ the attainable set at time T contains points of the form $(0,0,0,-CT^m + o(T^m))$. The result then follows from well-known sufficient conditions for STLC and a standard argument using convex approximating cones.

Start with fixing a control $\bar{u}\colon [0,T] \to [-1,1]$ (for some $T > 0$), such that $x_1(T,\bar{u}) = x_3(T,\bar{u}) = 0$ and $x_2(T,\bar{u}) > 0$.

We denote by $\bar{u}^{-1}$ the time-reversed control (defined by $\bar{u}^{-1}(t) = \bar{u}(T-t)$), and inductively define via concatenation $\bar{u}_1 = \bar{u}^{-1} * \bar{u}$ and $\bar{u}_k = \bar{u}^{-1} * \bar{u}_{k-1} * \bar{u}\colon [0,2kT] \to [-1,1]$. Finally for any given $t_0, \varepsilon_0 > 0$ let $\delta = \delta(k) = t_0/(2kT)$ and $u_k\colon [0,t_0] \to [-\varepsilon_0,\varepsilon_0]$, $u_k(\delta t) = \varepsilon_0 \bar{u}_k(t)$. One easily verifies $x_i(T,u_k) = 0$ for $i = 1,2,3$ and

$$x_4(T,u_k) = \varepsilon_0^6 t_0^9 k^{-8} C_{41} - \varepsilon_0^7 t_0^{15} k^{-7}(C_{42} + O(1/k))$$

with constants $C_{41}, C_{42} > 0$ depending on the initial choice of $\bar{u}$ only.

Thus for k sufficiently large, i.e. $k = k(\varepsilon_0, t_0) = K\varepsilon_0^{-1}t_0^{-6}$, we obtain

$$x(t_0,u_k) = (0,0,0,-Ct_0^{57} + o(t_0^{57}))$$

($C > 0$ and $K > 0$ are constants).

To prove Claim 2, let $u\colon [0,T] \to [-\varepsilon,\varepsilon]$ be such that $x_1(\cdot,u)$ changes sign at most $N-1$ times, $x_1(T,u) = 0$ and $N^7T^{7/2}\varepsilon^{3/4} < 1$. Choose times $0 = t_0 \leq t_1 \leq \cdots \leq t_r = T$ ($r \leq N$), such that $x_1|_{[t_j,t_{j+1}]}$ is of constant sign, $j = 0,1,\ldots,r-1$. We write $A = \int_0^T x_2^7(s,u)\,ds$ and we may assume $0 < A < 1$.

Find $T_1 \in (0,T)$ such that $x_2(T_1,u) > (A/T)^{1/7} = B$. Thus $x_2(\cdot,u)$ increases by at least $C = B/N$ on at least one subinterval $I_{j_0} = [t_{j_0}, T_{j_0+1}]$. Since $x_1(\cdot,u)$ is nonnegative on I_{j_0}, we may use the Hölder inequality without having to introduce absolute values and thus may conclude

$$x_3(t_{j_0+1},u) - x_3(t_{j_0},u) \geq C^3/T^2 = D.$$

W.l.o.g. we may assume $x_3(t_{j_0},u) \leq -\frac{1}{2}D < 0$ (else $x_3(t_{j_0+1},u) \geq +\frac{1}{2}D > 0$ leads to similar calculations), and using $|u(\cdot)| \leq \varepsilon$ and $x_1(t_{j_0}) = 0$ we know $x_3(t_{j_0}+s,u) \leq -\frac{1}{2}D + \frac{1}{4}\varepsilon^3 s^4$ for $0 \leq s \leq s_0 = (2D\varepsilon^3)^{1/4}$ and finally

$$\int_0^T x_3^2(s,u)\,ds \geq \int_0^{s_0} (-\tfrac{1}{2}D + \tfrac{1}{4}\varepsilon^3 s^4)^2\,ds \geq \left(\frac{A}{T}\right)^{27/28} N^{-27/4}\varepsilon^{-3/4}T^{-9/2} > A,$$

which finishes the proof of Claim 2.

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