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TOPOLOGICAL TRANSVERSALITY HOLDS IN ALL DIMENSIONS

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Moving one submanifold to be transverse to another is a basic and essential operation in the study of manifolds. In the differentiable and PL categories the proof that this operation is always possible is straightforward, and one of the first objectives in developments of the subject. In the topological category it is more difficult; the first progress came in the profound work of Kirby and Siebenmann [**4**]. Most of what they left unsettled was completed in [**7**], but a few resistant 4-dimensional cases have remained open. Here we sketch a proof of these exceptional cases; details will appear in Chapter 9 of [**3**]. We also indicate modifications required in the proof of [**4**] to reduce the theorem to these cases. In particular this includes the intersection dimension 4 case, which was claimed without proof in [**7**].

THEOREM. *Suppose M and X are proper submanifolds of Y, X has a normal microbundle ξ, there are closed subsets $C \subset D \subset Y$, and M is transverse to ξ near C. Then there is an isotopy of M supported in any given neighborhood of $(D - C) \cap M \cap X$, to a submanifold transverse to ξ near D.*

A submanifold is "proper" if closed, and the intersection with the boundary is the boundary of the submanifold. Transversality to a microbundle ξ means that the intersection $M \cap X$ is a manifold with a normal microbundle in M, and this microbundle is the restriction of ξ (see [**4**, III §1]). It is necessary to specify a bundle because the theorem is false for purely local versions of transversality; see [**4**, III §1]. Microbundle transversality implies transversality with respect to other bundle theories, e.g., [**6**].

Kirby and Siebenmann [**4**, p. 91] proved this assuming $\dim M \neq 4 \neq \dim M \cap X$, and either $\dim X \neq 4 \neq \dim Y$ or $\operatorname{codim} M \geq 3$. [**7**, Theorem 2.4.1] gave the remaining cases except when $\dim Y = 4$ and one of M, X has dimension greater than 2.

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SKETCH OF THE PROOF, FOLLOWING [**4**, p. 90]. The bounded case follows from two applications of the case with ∂Y empty. The "strongly relative" nature of the theorem allows reduction to ξ trivial, M with one coordinate chart and Y open in a product $M \times \mathbf{R}^q$. At this point Y has a PL structure as a subset of $M \times \mathbf{R}^q$, and M is PL in Y. The plan is to make X PL with respect to such a structure, and then use the PL transversality theorem.

PRODUCT STRUCTURE THEOREM. *Suppose $N^n \times \mathbf{R}^r$ has a PL structure, which near $W \times \mathbf{R}^r$ is the product with a PL structure on a neighborhood of $W \subset N$. Suppose that the closure of $N - W$ has no compact components if n is 3 or 4. Then there is a PL structure on N extending the one near W, so that the product structure on $N^n \times \mathbf{R}^r$ is concordant rel $W \times \mathbf{R}^r$ to the given one, and is isotopic except possibly when $n + r = 4$ and $n \geq 2$.*

REFERENCES. If $n \geq 5$, or $n + r \geq 5$ and $n \leq 3$, then this comes from [**4**, I §1]. If $n + r \leq 3$ the result follows from the existence and uniqueness of triangulations in these dimensions.

If $n = 4$ then the 3-connectivity of $\mathrm{TOP}(4)/\mathrm{PL}(4) \to \mathrm{TOP}/\mathrm{PL}$ [**7**, **5**], and noncompactness of N implies that the tangent microbundle of N has a PL structure compatible with the PL structure on the product. Immersion theory (see [**4**, p. 226]) shows this microbundle structure comes from a structure on the manifold, and implies the resulting product structure is concordant to the original. Since $n + r \geq 5$ the Concordance Implies Isotopy theorem [**4**] shows they are isotopic.

If $n+r = 4$ the concordance statement again comes from immersion theory and the connectivity of $\mathrm{TOP}(4)/\mathrm{PL}(4) \to \mathrm{TOP}/\mathrm{PL}$. The isotopy assertion when $n \leq 1$ comes from [**7**], see [**8**, Theorem 2.1].

This statement is false in the excluded cases. 3-manifolds have unique PL structures, while the high-dimensional theory predicts two on compact ones. The theorem therefore works about half the time in this case, and the obstruction is well understood. In dimension 4, PL structures behave rather strangely, and presently little useful can be said about "obstructions" (see [**8**] or [**3**, Chapter 8] for discussions).

PROOF (CONTINUED). Let L_0 denote a neighborhood of $C \cap M \cap X$ in $M \cap X$, so that M is transverse to ξ in a neighborhood of the closure of L_0. The first step is to arrange L_0 to be PL in M. We may arrange that L_0 has no compact components simply by deleting points from Y. A neighborhood in M is of the form $L_0 \times \mathbf{R}^n$ (by transversality, and triviality of ξ). According to the product structure theorem there is a concordance of the structure on M to one in which L_0 is a PL flat submanifold of $M - (M \cap X - \mathrm{int}\, L_0)$. No cases are excluded at this point.

The next step [**4**, p. 93] involves a division according to the codimension of M. When codim $M \geq 3$ the given proof applies without change, and completes these cases.

When codim $M \leq 2$ there is a normal bundle for L_0 in X. The case with dim $L_0 = 2$, dim $X = 4$ is excluded in [**4**], but is now known [**2**]. Extend this to a normal bundle in Y for a neighborhood of L_0 in M, using [**4**, Lemma 1.9, p. 99]. Extend this further to a normal bundle for $M \subset Y$ using the relative version of existence for normal bundles. Since we have reduced to M flat in Y this bundle must be trivial.

Now consider the intersection of X with this normal bundle for M. There is a PL structure on a neighborhood of L_0 in X which is a PL submanifold of Y and PL transverse to M. Since X has a product neighborhood $X \times \mathbf{R}^n$ in Y, the product structure theorem implies that, if dim $Y \neq 4$, there is an isotopy of X rel L_0 to a PL submanifold of Y.

The PL transversality theorem applies to give a further isotopy rel L_0 to make X transverse to M. Inverting these isotopies gives an isotopy which makes M transverse to X, as required for the theorem.

EXCEPTIONAL CASES. Cases with dim $Y = 4$ and dim X, dim $M \geq 2$ are excluded in the last step because the product structure theorem only gives a concordance. The case dim X = dim Y = 2 is proved in [8]. Roughly, M is arranged to be PL as above, and X is changed by *regular homotopy* to be PL, and then transverse. An isotopy is obtained by working back past the singularities of the regular homotopy.

In the remaining case we assume dim $X = 3$. The first step is to refine the initial reduction to reduce consideration to $X = \mathbf{R}^3$, $Y = D = \mathbf{R}^4$, and $C = (\mathbf{R}^3 - \text{int } D^3) \times \mathbf{R}$.

Choose a triangulation of X fine enough so that $X \cap C$ can be replaced by a subcomplex. Now proceed by induction (for 4 steps): suppose M is transverse to a neighborhood of $C \cup X^i$, and show it can be made transverse to a neighborhood of $C \cup X^{i+1}$. The induction step requires extending transversality from near the boundary of an $(i+1)$-simplex to a neighborhood of the entire simplex. Note that if M is transverse to the simplex itself then minor adjustment makes it transverse to X in a neighborhood of the simplex. But if $i+1 < 3$ then transversality to the simplex is a lower-dimensional problem, which we may assume already known. The remaining problem is the induction step for 3-simplices; $X = \mathbf{R}^3$ as above.

Recall that although $Y \simeq \mathbf{R}^4$, it is to be given a PL structure as a subset of $M \times \mathbf{R}^q$. This may be different from the standard structure on $\mathbf{R}^4$. X has been arranged to be PL with respect to this structure near $M \cap (\mathbf{R}^3 - \text{int } D^3)$. Next approximate X by a PL submanifold; choose a PL codimension 1 X_4 disjoint from, and in a neighborhood of, X, so that the intersection with M near $\mathbf{R}^3 - \text{int } D^3$ is parallel (in M) to the intersection with X. We also assume X_4 transverse to M, and discard any compact components.

Denote the region between X and X_4 by W. Since W is noncompact it has a handlebody structure; a collar on X union handles. Using the "parallel" condition it can be arranged that near $\mathbf{R}^3 - \text{int } D^3$ only the collar intersects M, and intersects in a subcollar. Consequently M intersects only finitely many of the handles of W.

Let X_i denote the level between the i and $(i+1)$-handles. Then X_i is obtained from X_{i-1} via i-handles, and dually X_{i-1} is obtained from X_i via $(4-i)$-handles. (This is "ambient surgery" on X_i). If X_i is transverse to M, and the cores of these dual handles are also transverse to M, then (as in the simplicial reduction above) M can be adjusted to be transverse to the "handlebody", and therefore to X_{i-1}. Transversality to the cores is a lower-dimensional problem as long as $4 - i < 3$, so can be assumed to be possible. This means we can make X_1 transverse to M.

The 0-handles in W can be easily cancelled with 1-handles, so $X = X_0$. The conclusion is that by changing X using finitely many 1-handles, it can be made transverse to M.

Now since $Y = \mathbf{R}^4$ and $X = \mathbf{R}^3$ (topologically), the 1-handles are homotopically trivial. Since homotopy implies isotopy for 1-manifolds, they are topologically trivial. In particular there are cancelling 2-handles, so it is also possible to retrieve X from X_1 using only 2-handles. But 2-handles are a lower-dimensional case, so as above M can be made transverse to the cores, and therefore to X.

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