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Foundations of algebraic analysis, by Masaki Kashiwara, Takahiro Kawai, and Tatsuo Kimura. Translated by Goro Kato. Princeton Mathematical Series, vol. 37, Princeton University Press, Princeton, 1986, xii + 254 pp., \$38.00. ISBN 0-691-08413-0

"Algebraic analysis" is a term coined by Mikio Sato. It encompasses a variety of algebraic methods to study analytic objects; thus, an "algebraic analyst" would establish some properties of a function or a distribution by investigating some linear partial differential operators which annihilate it. Here is a concrete example: let f be a polynomial in n complex variables $x_1, x_2, \ldots, x_n$; if s is a complex number, with $\mathrm{Re}(s) > 0$, $|f|^s$ is a well-defined continuous function on $\mathbf{C}^n$. Bernstein [**2**] showed that $|f|^s$ extends to a meromorphic function of s with values in distributions on $\mathbf{C}^n$. The key step is the abstract derivation of the following equation:

$$\text{(B-S)} \qquad P\left(x, \bar{x}, \frac{\partial}{\partial x}, \frac{\partial}{\partial \bar{x}}\right) |f|^s = b(s) \cdot |f|^{s-2},$$

where $b(s)$ is a nonzero polynomial in s, and P is a partial differential operator with polynomial coefficients involving both the variables x_i and their complex conjugates $\bar{x}_i$. This gives immediately the desired meromorphic continuation, with poles located at $\lambda - 2, \lambda - 4, \ldots$, for λ a zero of the polynomial b.

(B-S) is the so-called Bernstein-Sato differential equation; $b(s)$, if chosen of the form $s^k + a_{k-1}s^{k-1} + \cdots + a_0$ with k minimal, is the *Bernstein-Sato polynomial.* This polynomial is in general hard to compute [**14**, **20**]; its zeroes are related to the singularities of the hypersurface $f = 0$, in particular to the monodromy action on the vanishing cycles [**14**].

The algebraic ideas behind this theorem of Bernstein, as well as subsequent generalizations by Kashiwara [**10**] and Björk [**3**], involve so-called "D-modules", which means: modules over the algebra D of partial differential operators (usually, in the complex domain). For instance, the vector spaces of holomorphic (or meromorphic) functions, or of generalized functions, are D-modules, because differential operators operate on them in the usual way. Inside a huge D-module like the space of generalized functions, one may consider smaller ones. For instance, if h is a generalized function, the space of all $P(h)$, $P \in D$, is a D-module: the D-module generated by h. The basic idea now is that this D-module is of "small size" iff there are many P such that $P(h) = 0$, i.e. there are many linear PDEs satisfied by h. Algebraic analysis includes, in particular, many techniques to show that a given D-module is of "small size".

However, D-modules can never be that small! For instance, the algebra D for the complex plane $\mathbf{C}$ is generated by x and d/dx, with the famous relation $[d/dx, x] = 1$. Now it is easy to see that if a D-module M is nonzero, it has to be *infinite-dimensional.* Indeed, if it were finite-dimensional, by linear algebra, there would be a nonzero polynomial $P(x)$ which annihilates M. Then $[d/dx, P(x)] = P'(x)$ also annihilates M. Iterating the process, one finds a nonzero constant which annihilates M, hence M must be reduced to 0.

This phenomenon was first observed by Bernstein [**1**], who introduced the *dimension* of a D-module M, for $D = C[x_1, \dots, x_n, \partial/\partial x_1, \dots, \partial/\partial x_n]$ the algebra of partial differential operators on $\mathbf{C}^n$, with polynomial coefficients. He showed that a nonzero module has dimension at least n. This striking observation received the following geometric explanation [**18**]: To M one may associate a *characteristic variety* $\mathrm{Ch}(M)$, a subvariety of the *cotangent bundle* of $\mathbf{C}^n$. Sato, Kashiwara and Kawai prove that $\mathrm{Ch}(M)$ is an involutive subvariety; it follows that its dimension is at least n.

When $\mathrm{Ch}(M)$ has dimension exactly n, it is a *lagrangian* variety. Then M is called a *holonomic D-module.* This class of D-modules has remarkable stability properties, and so have the (generalized) functions which generate a holonomic D-module. For instance, $|f|^s$, for given $s \in \mathbf{C}$, is of this type; so is the character of an irreducible unitary representation of a semisimple Lie group. The theory of holonomic D-modules is essentially due to Kashiwara [**8, 9**]. The more difficult theory of holonomic D-modules with regular singularities was developed by Mebkhout [**16**] and by Kashiwara-Kawai [**11**], and has found many applications to algebraic geometry, representation theory, and even number theory, for which one may refer to the excellent expository articles [**12** and **17**].

The reason the Kyoto school of Sato was so interested in analyzing D-modules inside the cotangent bundles (where their characteristic varieties sit) was because of Sato's intuition that the singularities of an analytic object (say a distribution, or, perhaps more appropriately, a hyperfunction) propagate inside the cotangent bundle. This is the well-known principle of *microlocalization*, which also underlies Hörmander's concept of wavefront set of a distribution.

The notion of *hyperfunction* is central in the field of algebraic analysis. For Sato, a hyperfunction on an open set U of $\mathbf{R}^n$ is a *boundary value* of a holomorphic function defined on $U + \sqrt{-1} \cdot C$, where C is an open cone in $\mathbf{R}^n$. For instance, $1/(x \pm i \cdot 0)$ is the hyperfunction on $\mathbf{R}$ which is the boundary value of the holomorphic function $1/x$ in $\mathbf{R} \pm \sqrt{-1} \cdot \mathbf{R}$. The equality

$$\delta(x) = \frac{1}{2\pi i}\left[\frac{1}{x + i \cdot 0} - \frac{1}{x - i \cdot 0}\right],$$

which is a proposition in the classical book of Gel′fand and Shilov [**5**, Chapter I, §3], is taken as the definition of the Dirac δ-function by Sato. Hyperfunctions are equally interesting as distributions, especially where systems of linear PDEs with *real-analytic coefficients* are concerned. They are more concrete objects, and they are also more geometric in nature, since they are intrinsically related to complex analysis. There is another point of view on hyperfunctions, in which they are linear functionals on analytic functions; this is due to Martineau [**15**].

Sato and his school developed a powerful calculus for hyperfunctions, based on some beautiful topological constructions, the most important of which is the notion of *singularity spectrum* of a hyperfunction $u(x)$, denoted $\mathrm{SS}\, u(x)$, which is a conical subset of the cotangent bundle T^*M. An essential property of $\mathrm{SS}\, u(x)$ is that it is contained in a closed convex set $Z \subset T^*M$ iff there is a conical neighborhood V of the polar set $Z^0 \subset TM$ such that $u(x)$ is the boundary value of a holomorphic function defined in V (in this sentence, when we say $Z \subset T^*M$ is convex, we mean that its fibre over each point of M is convex). The singularity spectrum is the analog, for hyperfunctions, of Hörmander's wavefront set. This motivates the definition of *microfunctions*: these objects, defined microlocally (i.e., locally on T^*M) are defined, on some open set U, as hyperfunctions modulo those whose singularity spectrum does not meet U (such hyperfunctions being, so to speak, invisible to microscopic observation inside U).

In algebraic analysis, sheaf-theoretic techniques are essential even at the very beginning of the theory. Indeed, a fundamental insight of Sato is to define hyperfunctions on a manifold M as elements of the vector space $H^n_M(X, \mathcal{O}_X)$, a cohomology group of a complexification X of M, with values in the sheaf $\mathcal{O}_X$ of holomorphic functions, and *with support* in the subspace M. Such groups are well known to modern-day algebraic geometers, but they still are quite unfamiliar to most analysts. I think this is one reason why algebraic analysis has had relatively little influence outside of Japan, France, and the Soviet Union. To make things worse, for many years the only widely available source on the subject was the famous 264-page article of Sato, Kashiwara, and Kawai [**18**], the aridity of which discouraged almost every reader. The situation should change now, with the book under review available in English translation. Many mathematicians will want to learn the theory from this remarkable book. They will be immensely rewarded by the many applications, especially to systems of linear PDEs, several of which are described in the second half of the book (I will comment on these applications later).

The book is a complete and self-contained introduction to the theory of hyperfunctions and microfunctions. Since these are defined using sheaf cohomology, the book begins with a 21-page minicourse on sheaf theory. It will probably be exciting for the analyst who wants to learn about sheaf theory to see how naturally it applies to boundary values of holomorphic functions. The topological ideas in Chapter II, §3, are hard to understand, and are quite deep (they are related to topological Fourier and Radon transforms). Theorem 2.3.5 gives the geometric property of the singularity spectrum, mentioned previously. After 80 pages of hard theory, II, §4 gives a large number of interesting examples, some of which are quite subtle.

Chapter III presents the main operations on hyperfunctions and microfunctions: products (on different manifolds), restriction (to a submanifold which is noncharacteristic with respect to the singularity spectrum), product of two hyperfunctions on the same manifold (when it makes sense!), and integration. Each operation is illustrated by interesting examples. III, §3 contains a discussion of Feynman integrals and their analyticity which has rather confused me. The next paragraph is very important because it introduces *microlocal operators* (which operate on the space of microfunctions on M) as *microfunction kernels* on $M \times M$, supported, in some sense, on the diagonal. It also gives a proof of a celebrated theorem of Sato, which says that an elliptic microlocal operator is invertible. The next two sections contain a theorem of Holmgren (on the Cauchy problem for hyperfunctions), and the construction of fundamental solutions of the wave equation and of general hyperbolic equations. The "watermelon theorem" is stated but not proven.

The book culminates in the fourth chapter, on microdifferential operators, which is very beautiful. Unfortunately, §1, which introduces the operators and establishes various formulae, is very hard to follow in detail, because the constructions use Radon transformation, instead of Fourier transformation (cf. for instance [**13**]). Thus, the formula for the product of two microdifferential operators is obtained by a devious route, which hides its fundamental simplicity (see [**7**]). It will be useful for the reader to keep in mind that these operators, first introduced by Boutet de Monvel and Kree [**4**], are the natural complex-analytic analogs of pseudodifferential operators.

Important theorems of Späth and Weierstrass type are stated, but not proven. The rest of the chapter is extremely interesting and much easier to follow. §2 explains quantized contact transformations (with only one example; more can be found in [**9**]). The last paragraph presents the most accessible applications to linear PDEs. A typical theorem (4.3.2) states that a microdifferential operator of order 1 with the same principal symbol as $\partial/\partial x_1$ is microlocally equivalent to it. A more sophisticated result is a microlocal normal form for a microdifferential operator such that its principal symbol f is zero at some point, but such that the Poisson bracket $\{f, \overline{f}\}$ is greater than 0 at that point. The most spectacular theorem gives a microlocal interpretation of H. Lewy's discovery of equations without local solutions.

Many people, after reading this book, will want to learn more about the applications to PDEs. They may look into recent works by Kashiwara, Kawai, Schapira, and many others. I will suggest the books [**6** and **19**].

The book is overall quite readable, with some exceptions I have already pointed out. There are, however, rapid fluctuations, inside the book, in the level of difficulty; so skipping difficult parts at first might be a good idea. I have to criticize the systematic quotation of textbooks *in Japanese* for reference on some standard facts (e.g., on p. 26, the reader is referred to a book by Hitotumatu for a theorem of Grauert). The translation is generally very good, but some details could be improved. The American reader will probably be surprised by the occasional use of the word "exquisite" about mathematical theorems (see p. 140). The style of the translation makes III, §3, hard to read.

The purpose of the authors, stated in their introduction, to write a "Courant-Hilbert" for the new generation, might be a trifle too ambitious. Nevertheless, by this beautiful and masterful introduction to algebraic analysis, they will certainly contribute to the dissemination of these rich and powerful ideas to a wide mathematical audience.

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Recursively enumerable sets and degrees: A study of computable functions and computably generated sets, by Robert I. Soare. Perspectives in Mathematical Logic, Springer-Verlag, Berlin, Heidelberg, New York, 1987, xviii + 437 pp., $35.00. ISBN 3-540-15299-7

One of the tantalizing aspects of twentieth century mathematical logic is the juxtaposition of the highly theoretical with the very practical. Logical investigations, aimed at giving precise mathematical definitions of "theorem", "proof", and "mathematical truth", led naturally to a study of computable processes. As a result, it is now generally recognized that recursive function theory provides the theoretical foundation for computer science.