it is a pity that the reprint apparently has not been used to make corrections, because there are many minor, but sometimes irritating, misprints.

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Semigroups of linear operators and applications, by Jerome A. Goldstein. Oxford Mathematical Monographs, Oxford University Press, Clarendon Press, New York and Oxford, 1985, $x + 245$ pp., $42.50. ISBN 0-19-503540-2

This year is the centenary of the founding of the analytic theory of one-parameter semigroups: In 1887 Giuseppe Peano [**Pe**] wrote the system of ordinary differential equations

$$\begin{aligned} du_1/dt &= a_{11}u_1 + \cdots + a_{1n}u_n + f_1(t) \\ &\ \vdots \\ du_n/dt &= a_{n1}u_1 + \cdots + a_{nn}u_n + f_n(t) \end{aligned}$$

in matrix form as $du/dt = Au + f$ and found the explicit formula

$$u(t) = e^{tA}u(0) + \int_0^t ds\, e^{(t-s)A} f(s)$$

for the solution, where $e^{tA} = \sum_{k=0}^{\infty} t^k A^k/k!$. The mapping $t \geq 0 \to T(t) = e^{tA}$ is called the semigroup generated by A. More generally a C_0-semigroup on a Banach space $\mathbf{X}$ is a strongly continuous mapping from $\mathbf{R}_+$ into the bounded operators on $\mathbf{X}$ with the properties $T(t+s) = T(t)T(s)$ and $T(0) = 1$. The generator of T is the operator A defined by $Af = \lim_{t\to\infty}(T_t f - f)/t$ where f is in the domain $D(A)$ of A if and only if the limit exists. These concepts were introduced by Hille in the thirties, and he studied the semigroup by means of the resolvent $(\lambda - A)^{-1} = \int_0^\infty dt\, e^{-\lambda t}T(t)$. A fundamental theorem, proved by Hille and Yosida for contraction semigroups, and Feller, Miyadera, and Phillips for general semigroups around 1950, states that A is the generator of a C_0-semigroup T if and only if A is a closed, densely defined operator, and there exist real constants M, ω such that the resolvent $(\lambda - A)^{-1}$ exists for $\lambda > \omega$ and

$$\|(\lambda - \omega)^n (\lambda - A)^{-n}\| \leq M$$

whenever $\lambda > \omega$ and $n = 1, 2, 3, \ldots$. In this case $\|T(t)\| \leq Me^{\omega t}$, and

$$T(t)f = \lim_{n\to\infty} \left(1 - \frac{t}{n}A\right)^{-n} f \quad \text{for all } f \in \mathbf{X}.$$

As already Peano's example shows, semigroups turn up when one wants to solve the Cauchy problem

$$du(t)/dt = Au(t), \qquad t \geq 0,\ u(0) = f,$$

where A is a closed densely defined operator on $\mathbf{X}$. Typically $\mathbf{X}$ is a function space and A is a differential operator. The Cauchy problem is said to be well posed if the resolvent set of A is nonempty and for each $f \in D(A)$ there is a unique solution $u\colon \mathbf{R}_+ \to D(A)$ of the Cauchy problem such that $t \to u(t)$ is continuously differentiable. A fundamental result of Hille and Phillips is that the Cauchy problem is well posed if and only if A generates a C_0-semigroup T on $\mathbf{X}$, and then the unique solution is given by $u(t) = T(t)f$; see [**HP**]. The interest of semigroup theory in connection with differential equations stems from this result, and semigroup methods were used to study parabolic partial differential equations and scattering theory in the fifties and sixties by A. Friedman, E. Nelson, E. Dynkin, G. Hunt, P. Lax, J. Moser, R. Phillips, T. Kato and others.

Since then, a considerable number of books, including the present one, have been written on semigroup theory and its applications to differential equations; a selection is [**B-M**, **F1**, **F2**, **Fr**, **G**, **K1**, **K2**, **Kr**, **Pa**, **T**, **Y**, **Z**]. Although the viewpoint of the present book is similar to that of many of the others, it differs by having a somewhat more encyclopaedic character. Also, as the author himself states, the emphasis is on motivation, heuristics and applications. In addition to the main results, the book contains many specialized results which are stated without proofs. The book makes short forays into fields of applications other than differential equations, such as random evolutions, control theory, functional differential equations, the Feynman path formula, and Feller-Markov processes. Here are a couple of samples:

Let T_n be a sequence of C_0-semigroups, uniformly bounded in n and with generators A_n, and let T be a C_0-semigroup with generator A. Then it is known from the Kato-Trotter-Kurtz theorem that $\lim_{n\to\infty} T_n(t)f = T(t)f$ for all $t \geq 0$, $f \in \mathbf{X}$, uniformly for t in compacts, if and only if A is the graph limit of A_n. (This version of the theorem seems to have been overlooked by a large number of authors, including the present one; see [**BrR**, Theorem 3.1.28 or **D**, Theorem 3.17].) This can be used to give a simple proof of the classical Weierstrass approximation theorem: Let $\mathbf{X}$ be the Banach space of bounded uniformly continuous functions on $\mathbf{R}$ and define $(T(t)f)(x) = f(t+x)$ for $f \in \mathbf{X}$. Then T is a contraction semigroup with generator $A = d/dx$. Put $A_n = (T(1/n) - 1)/(1/n)$. It follows from the Kato-Trotter-Kurtz theorem that

$$f(t) = \lim_{n\to\infty} \lim_{M\to\infty} \sum_{m=0}^{M} t^m (A_n^m f)(0)/m!$$

uniformly for t in compacts, which gives Weierstrass's theorem. (This example is due to Hille.)

Another interesting proof of a classical theorem is due to the author himself: A useful theorem of Chernoff [**C**] states that if $V(t)$, $t > 0$, is a family of contractions on $\mathbf{X}$ with $V(0) = 1$ such that the derivative $V'(0)f$ exists for all f in a dense space D, and the closure A of $V'(0)$ generates a C_0-contraction semigroup T, then

$$\lim_{n\to\infty} V(t/n)^n f = T(t)f$$

for each $f \in \mathbf{X}$, uniformly for t in compacts. If ξ is a random variable, let F_ξ be its distribution function (i.e. $F_\xi(x)$ is the probability that $\xi \le x$) and let

$$F_{0,t}(x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{x} ds\, e^{-s^2/2t}$$

be the distribution function of a gaussian random variable with mean 0 and variance t. The central limit theorem states that if $\xi_1, \xi_2, \ldots$ is a sequence of independent identically distributed random variables with mean 0 and variance 1, then

$$\lim_{n\to\infty} F_{(1/\sqrt{n})\sum_{k=1}^{n}\xi_k}(x) = F_{0,1}(x)$$

for all $x \in \mathbf{R}$. This can be proved as follows: Any distribution function F defines a bounded operator $\tilde{F}$ on $\mathbf{X} = C_0(\mathbf{R})$ by convolution $\tilde{F}f(x) = \int_{-\infty}^{\infty} f(x-s)\, dF(s)$. It suffices to show that

$$\|\tilde{F}_{(1/\sqrt{n})\sum_{k=1}^{n}\xi_k} f - \tilde{F}_{0,1} f\| \to 0$$

for all f in $\mathbf{X}$. Let G be the common distribution function of all the ξ_k, and define $G_t(x) = G(x/\sqrt{t})$, $V(t) = \tilde{G}_t$ for $t > 0$, and $V(0) = 1$. Let $A = \frac{1}{2}(d^2/dx^2)$. It is well known that A generates the heat semigroup, i.e., the semigroup given by $T(t) = \tilde{F}_{0,t}$ for $t > 0$. Now one can show that

$$\lim_{t\to 0} \|(V(t)f - f)/t - Af\| = 0$$

provided $f \in D(A)$.But then by Chernoff's theorem

$$\lim_{n\to\infty} \|V(t/n)^n f - \tilde{F}_{0,t} f\| = 0$$

for $f \in \mathbf{X}$. Now, put $t = 1$ and use that

$$V(1/n)^n = (\tilde{G}_{1/n})^n = (G_{1/n} * \cdots * G_{1/n})^{\sim},$$

where $G_{1/n} * \cdots * G_{1/n}$ is the distribution function of

$$\frac{\xi_1}{\sqrt{n}} + \frac{\xi_2}{\sqrt{n}} + \cdots + \frac{\xi_n}{\sqrt{n}},$$

to derive the central limit theorem.

The encyclopaedic character of the book is further enhanced by a large number of exercises, historical remarks and a reference list with about 1500 entries. Except for a few remarks and a treatment of the Navier-Stokes equation, the theory of nonlinear semigroups is omitted, but the author announces that this will be the subject of a subsequent volume. Another subject which is only mentioned in passing is the recent developments in the theory of positivity-preserving semigroups [**ACK**, **BaR**, **Br**, **BrJ**, **N**]. In case the richness of applications in Goldstein's book should give some first-time students of the subject indigestion, we mention in the references some books which give a more abstract introduction to semigroup theory. For more mature students and researchers, including engineers and scientists, who want to get an overview of the subject, this is a very useful book.

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