that if $f \in W_m^p(\Omega)$, where $\Omega \subseteq \mathbf{R}^n$ is a domain that satisfies a so-called cone condition, and $1 < p < n/m$, then $f \in L^{p^*}(\Omega)$, where $1/p^* = 1/p - m/n$. If $p > n/m$, then $f \in C(\Omega)$, and if $p = n/m$, then $f \in L^q(\Omega)$ for all $q < \infty$. (The theorem was later extended to $p = 1$ by L. Nirenberg and E. Gagliardo.)

Generalizations and refinements of this theorem are one of the main themes of the book under review. The author is especially interested in finding necessary and sufficient conditions on the domain Ω for the validity of various embedding theorems. For $p = 1$ such conditions can be given in terms of isoperimetric inequalities, relating the volume and the surface area of portions of the domain. For $p > 1$ the area has to be replaced by "p-capacity".

Another type of extension theorem is obtained if the domain is allowed to be all of $\mathbf{R}^n$, but embeddings into spaces $L^q(\mu)$ are considered for positive measures μ. The measures allowing such embeddings are characterized in terms of capacities. These results are in part due to D. R. Adams.

An interesting and useful chapter, written jointly with Yu. D. Burago, treats spaces of functions of bounded variation, i.e., functions whose derivatives are measures.

Generally speaking, this book is not the right choice for someone who is just trying to learn a few simple facts about Sobolev spaces. The author's taste is for completeness. He treats every conceivable aspect of his problems, which makes the book rather overwhelming for the general reader.

On the other hand, this makes the book all the more valuable as a work of reference. It is a treasure house, for example, for someone who is looking for a weird domain as a counterexample to some theorem, and for many others. Every good mathematical library should have it.

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Complex cobordism and stable homotopy groups of spheres, by Douglas C. Ravenel, Academic Press, Orlando, 1986. xix + 413 pp., \$90.00 cloth, \$45.00 paperback. ISBN 0-12-583430-6

The author has previously written [4, p. 407]: "I am painfully aware of the esoteric nature of this subject and of the difficulties faced by anyone in the past who wanted to become familiar with it." The subject in question is the topic of the book under review, namely the study of stable homotopy theory by means of the Adams-Novikov spectral sequence. Here, then, is required reading for all who want to enter this field.

The holy grail of stable homotopy theorists is the calculation of the stable homotopy groups of spheres, $\pi_k^S = \pi_{n+k}(S^n)$ for $n > k+1$ (the independence of these groups on n in this range follows from the suspension isomorphism). One finds easily that $\pi_0^S \cong \mathbf{Z}$ and $\pi_k^S = 0$ for $k < 0$. Serre proved in 1953 [**5**] that, for $k > 0$, π_k^S is a finite abelian group. Thus one can fix a prime p and ask to compute the p-primary components $\pi_{k(p)}^S$ for $k > 0$; the problem is usually approached in this form.

Why is the study of the stable homotopy groups of spheres such a central problem? We shall offer some remarks, and in addition urge the reader to consult G. Whitehead's lovely article [**7**] for an account of the development of homotopy theory.

The spheres are the most fundamental spaces in algebraic topology. There are many classical theorems about spheres, such as the Borsuk-Ulam theorem and the result of H. Hopf that the degree of a map of the n-sphere S^n to itself determines the homotopy class of the map. All graduate students in mathematics learn to compute the homology of S^n. Since it is even simpler to define the homotopy groups $\pi_{n+k}(S^n)$, as the homotopy classes of base-point preserving maps $S^{n+k} \to S^n$, it may come as a surprise to learn that these groups present such a challenge, even for the two- and three-dimensional spheres.

In the face of such a challenge, one looks for patterns. Perhaps the most impressive is that the suspension homomorphism

$$\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$$

is an isomorphism for $n > k+1$, proved by Freudenthal in 1937. This brings one to the stable homotopy groups of spheres, where Serre's result mentioned above is the most striking pattern one sees next. Cell complexes are built from spheres; for a finite cell complex, Serre's results imply that the groups $\pi_{n+k}(S^n X)$ $(n \gg k)$ are finitely generated with the same rank as $H_k(X; \mathbf{Z})$. It is often straightforward to calculate homology of cell complexes, but an inductive computation of the stable homotopy groups of a cell complex would require a knowledge of the stable homotopy groups of spheres.

Thus there is ample justification for the creation of elaborate machinery to study the stable homotopy groups of spheres. Such an apparatus has been erected, the building blocks being generalized homology theories (especially bordism theory) and various associated spectral sequences.

There are several reasons why people find this an attractive field. Some are keen on homotopy theory (always have been, always will be). Some thrive on a challenge and enjoy computing. And some are devotees of a part of the machinery which can be brought to bear on the problem, much of which (especially generalized homology such as bordism theory) has independent interest.

Thus it is fortunate or unfortunate, depending on one's outlook, that the machinery used to study the stable homotopy groups of spheres has considerable intricacy. The classical Adams spectral sequence [**1**] was developed for this purpose. Fixing a prime p and letting A denote the Steenrod algebra of cohomology operations in mod p cohomology, the E_2-term of this spectral sequence is

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\mathbf{F}_p, \mathbf{F}_p)$$

while the limit groups $E_\infty^{s,t}$ for $t - s = k$ form the associated graded group to a filtration of $\pi_{k(p)}^S$. It is a major task to determine the E_2-term, which one has been able to accomplish only in a limited range for each prime. Moreover, one still faces the task of determining the differentials $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$.

The main object of interest in Ravenel's book is the Adam-Novikov spectral sequence—its origin, its detailed structure, and its applications. Surprisingly, one is asked to replace ordinary homology and cohomology by complex bordism and cobordism theory. The complex bordism groups $MU_n(X)$ of a space X are obtained from maps $f\colon M \to X$ of closed smooth n-dimensional manifolds into X, M having a complex structure on its stable normal bundle, under the equivalence relation of bordism. The complex bordism groups $MU_*(X)$ form a graded module over $MU_* = MU_*(pt)$, and the latter is a polynomial ring $\mathbf{Z}[x_2, x_4, \dots]$, where x_{2n} is represented by a suitable $2n$-dimensional manifold (examples: $x_{2(p-1)} = [\mathbf{C}P^{p-1}]$ for each prime p). One also replaces the Steenrod algebra by $MU_*(MU)$, the "self-homology of MU" which is analogous to the dual of the Steenrod algebra, and obtains a spectral sequence converging to π_*^S with

$$E_2^{s,t} = \mathrm{Ext}_{MU_*(MU)}^{s,t}(MU_*, MU_*).$$

More economical for the study of the p-primary component $\pi_{*(p)}^S$ is the replacement of MU by Brown-Peterson homology BP, for which $BP_* = BP_*(pt)$ is a polynomial ring $\mathbf{Z}_{(p)}[v_1, v_2, \dots, v_n, \dots]$, v_n of degree $2(p^n - 1)$. Now the E_2-term is

$$\mathrm{Ext}_{BP_*(BP)}^{s,t}(BP_*, BP_*),$$

and the spectral sequence converges to $\pi_{*(p)}^S$.

What has one gained by replacing ordinary cohomology with complex bordism or Brown-Peterson homology? First, there are *fewer differentials.* The first two figures of the book, on pp. 12 and 15, compare the Adams spectral sequence with the Adams-Novikov spectral sequence for $p = 3$ and $t - s \leq 45$; the reviewer had some fun locating the four differentials inadvertently omitted from the first of these, and suggests that the reader try to spot them. Secondly, one obtains *a richer homology theory,* built from manifolds and universal among homology theories having a formal group. Formal groups $F(X, Y)$ are power series over a commutative ring with unit element, satisfying the group-like conditions $F(X, 0) = X$, $F(X, Y) = F(Y, X)$ and $F(X, F(Y, Z)) = F(F(X, Y), Z)$; they enter into various areas of algebraic geometry and number theory; e.g., elliptic curves have formal groups [**6**, Chapter IV] expressing the group law near the origin. A great deal of number theory enters into the study of bordism and cobordism via formal groups, and much insight has been gained in the process, notably in working out the ideas of Jack Morava [**3**].

In recent years, there has been a trend toward less computational themes in stable homotopy theory. This began with the study of infinite families and periodicity phenomena, for example in work by M. Mahowald and by

H. Miller, D. Ravenel, and W. S. Wilson in the 70s. More recently, notable success has been obtained in understanding the structure of the stable homotopy category. As an example, Nishida's nilpotence theorem, asserting that all elements of π_k^S with $k > 0$ are nilpotent, has been greatly extended in work of E. Devinatz, M. Hopkins, and J. Smith.

Here now is some advice to users of the book under review. The first chapter is an informal introduction to the field, and can be read for orientation. The next two chapters deal with generalities about Adams spectral sequences, and the use of the classical Adams spectral sequence. The heart of the book is to be found in Chapters 4 and 5, on the Adams-Novikov spectral sequence and the chromatic spectral sequence; the latter serves to organize and make understandable the structure of the former. To get the most out of this part of the book, the reader should be familiar with a bit more about Brown-Peterson homology than is offered here; see §2 of Chapter 4 or Wilson [**8**] for guidance. The appendices on Hopf algebroids and formal groups also provide essential information. The results in the final section of Chapter 6 are needed as input for the calculations of the final chapter.

There are quite a few figures and tables in the book; the reader might want to compile an index for them, as the reviewer did. It takes a certain amount of youthful zeal to tabulate the 5-primary groups $\pi_{*(5)}^S$ in dimensions up to 1000, the previous record being 760 by Aubry [**2**]. In contrast, a smooth picture of $\pi_{*(p)}^S$ follows from the Adams-Novikov spectral sequence in dimensions up to $2p(p^2-1)$ for odd primes (in dimensions up to 25 for $p = 2$). Thus calculations in dimensions above 25 for $p = 2$, and above 240 for $p = 5$, require a devotion to special techniques such as those presented in the final chapter.

Having spent several very intense weeks with this book I have considerable respect for the hard work and mastery that went into it. There is a good mix of explanation and demonstration of techniques in action. One need not study the entire book to profit from it.

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