like a noncommutative version of the Chern character. This opens up a whole new subject of "noncommutative differential geometry". Furthermore, the algebraic formalism of the behavior of the trace leads one to the theory of cyclic cohomology. "But that is the subject for another book **[Cn 3]**", as Blackadar says at the end of his final chapter. (If you can't guess what the "**[Cn 3]**" refers to then you will have to look it up in Blackadar's bibliography.)

Final verdict: this is an excellent book, combining formidable scholarship, impeccable accuracy, and lucid if succinct exposition. It sets a very high standard for Springer's commendable new series of MSRI Publications.

CHRISTOPHER LANCE

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 18, Number 1, January 1988

0273-0979/88 $1.00 + $.25 per page

Applications of Lie groups to differential equations, by Peter J. Olver. Graduate Texts in Mathematics, Volume 107, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1986, xxvi + 497 pp., $54.00. ISBN 0-387-96250-6

A standard introductory textbook on ordinary or partial differential equations presents the student with a maze of seemingly unrelated techniques to construct solutions. Usually these unmotivated and boring techniques constitute the total experience with differential equations for an undergraduate. Faced with a given differential equation which is not a textbook model, one is hopelessly lost without "hints"!

In the latter part of the 19th century Sophus Lie introduced the notion of continuous groups, now known as Lie groups, in order to unify and extend these bewildering special methods, especially for ordinary differential equations. Lie was inspired by lectures of Sylow given at Christiania, present-day Oslo, on Galois theory and Abel's related works. [In 1881 Sylow and Lie collaborated in editing the complete works of Abel.] He aimed to use symmetry to connect the various solution methods for ordinary differential equations in the spirit of the classification theory of Galois and Abel for polynomial equations. Lie showed that the order of an ordinary differential equation can be reduced by one if it is invariant under a one-parameter Lie group of point transformations. His procedures were both constructive and aesthetic.

For ordinary differential equations Lie's work systematically and comprehensibly related a miscellany of topics including: integrating factors, separable equations, homogeneous equations, reduction of order, the method of undetermined coefficients, the method of variation of parameters, Euler equations, and homogeneous equations with constant coefficients. Lie also indicated that for linear partial differential equations, invariance under Lie groups leads directly to superpositions of solutions in terms of transforms [**1**]. Why has Lie's approach not been adopted in standard textbooks?

A symmetry group of a system of differential equations is a group which maps solutions to other solutions of the system. In Lie's framework these groups consist of point transformations depending on continuous parameters, acting on the space of independent and dependent variables. Elementary examples include translations, rotations, and scalings. More generally, a solvable autonomous system of first-order ordinary differential equations essentially defines a one-parameter Lie group of transformations. Lie showed that, unlike discrete groups, for example reflections, the continuous group of point transformations admitted by a differential equation can be found by an explicit computational algorithm.

The applications of continuous groups to differential equations make no use of the global aspects of Lie groups. The applications use connected local Lie groups of transformations. Lie's three fundamental theorems showed that such groups are completely characterized in terms of their infinitesimal generators, which form a Lie algebra determined by its structure constants. Lie groups, and hence their infinitesimal generators, are naturally extended or "prolonged" to act on the space of independent variables, dependent variables and derivatives of the dependent variables. Consequently, the nonlinear conditions of invariance of a given system of differential equations under Lie groups of transformations reduce to linear homogeneous conditions in terms of the infinitesimal generators of the group. These conditions are called the determining (or defining) equations of the group. Since the determining equations form an overdetermined system of linear homogeneous partial differential equations, one can usually determine the infinitesimals in closed form. For a given differential equation the entire procedure to determine the infinitesimal generators of its invariance group is routine; symbolic manipulation programs [**2**] have been developed to implement the scheme.

Accomplishments in applying Lie groups to differential equations also include:

(1) The use of differential invariants, resulting from the infinitesimal (Lie group) approach, to find the most general differential equation invariant under a given symmetry group.

(2) If an ordinary differential equation is derivable from a variational principle through a Lagrangian or Hamiltonian formulation, then invariance under a one-parameter "variational" symmetry leads to a reduction of order by two.

(3) If a system of partial differential equations is invariant under a Lie group of transformations, one can find constructively special classes of solutions, called similarity or invariant solutions, which are invariant under some subgroup of the full group admitted by the system. These solutions correspond to a reduction in the number of independent variables. In the case of two independent variables, the reduction is to a system of ordinary differential equations. This use of Lie groups was initiated by Lie but first came to prominence in the late 1950s through the work of the Soviet group at

Novosibirsk, led by Ovsiannikov [3]. Others [4] have shown how to use invariance to construct similarity solutions for specific boundary value problems. Special cases of such solutions include self-similar solutions which are derivable from dimensional analysis or more generally through invariance under groups of scalings (cf. Birkhoff [5]), and traveling wave solutions which are derivable from invariance under translations. The connections between invariant solutions and separation of variables have been developed by Miller [6] and his co-workers.

(4) Without explicitly solving the determining equations for its infinitesimal generators, one can determine constructively whether or not a given nonlinear system of differential equations can be mapped into a system of linear differential equations by some one-to-one transformation. Moreover, one can find explicitly the transformation if such a mapping is possible. In general, Lie groups can be used to find mappings between equations provided the target equation can be characterized completely in terms of invariance under a Lie group of transformations. Most of the work in this direction has been developed in the past decade.

(5) In 1918 Emmy Noether [7] showed how the symmetry group of a variational integral (variational symmetry) leads constructively to a conservation law for the corresponding Euler-Lagrange equations. For example, conservation of energy follows from invariance under translation in time; conservation of linear and angular momenta respectively from translation in space and rotational invariances. Such variational symmetries leave invariant the Euler-Lagrange equations but the converse is false.

(6) Recently the use of generalized symmetries defined by infinitesimal generators, including derivatives of the relevant dependent variables, has extended further the applicability of Lie groups to differential equations. The possibility of the existence of such transformations (which, according to Olver, are mistakenly called Lie-Bäcklund transformations in the modern literature) was recognized by Noether in her celebrated paper and came to fruition in the works of Kumei [8] and other authors. These generalized symmetries cannot be represented in closed form from the integration of a finite system of ordinary differential equations as is the case for Lie groups. Generalized symmetries can be computed for a given differential equation by a simple extension of Lie's algorithm. They can be shown to account for the conserved Runge-Lenz vector for the Kepler problem and the infinity of conservation laws for the Korteweg-de Vries equation and other nonlinear partial differential equations exhibiting soliton behavior. Furthermore, multi-soliton solutions are similarity solutions for corresponding multiparameter generalized symmetries. The invariance of a partial differential equation under a generalized symmetry usually leads to invariance under an infinite number of generalized symmetries. The means of constructing an infinite number of such symmetries through the use of recursion operators was ingeniously shown by Olver in his famous paper [9].

The book by Olver. Olver's text, evolving from a set of lecture notes distributed at Oxford [10], is certainly the most scholarly and comprehensive book on the application of Lie groups to differential equations since the appearance of Lie's collected works. The book is carefully written and well organized, with many examples, numerous exercises (some qualifying as thesis projects), detailed introductions, and valuable, long historical notes for each of its seven chapters. There is an excellent seven-page Introduction, a useful introductory Notes to the Reader, a superb 18-page list of references, and separate symbol, author, and subject indices. Olver has many entertaining and delightfully provocative comments.

The reader should not be daunted by the somewhat detailed "introductory" chapter. Only the final two chapters on Hamiltonian methods for finite-dimensional systems and evolution equations require the full abstract machinery of the first chapter.

Chapter 2 is concerned with symmetry groups of differential equations. In addition to discussing the basic topics, Olver presents novel results on multi-parameter invariance of ordinary differential equations, overdetermined and undetermined systems, and solvable groups. In the notes of this chapter he provocatively states that "Needless to say, all the results stated here have many alternative restatements and reformulations, using more and more technical and abstract mathematical machinery, a pointless exercise enjoyed by a number of researchers. The net result, of course, is always the same no matter how one tries to dress it up; the unfortunate reader of these versions comes away thoroughly confused, learning nothing of the ease and efficacy of applying his theory to concrete problems."

The third chapter includes a seven-step methodology for determining the group admitted by a given system of differential equations, classification results for group-invariant solutions and a rather short account of dimensional analysis.

The fourth chapter gives a fine account of Noether's work. It includes a new result on the reduction by two of the order of the Euler-Lagrange equations from the invariance of the variational integral under a one-parameter variational symmetry. Olver shows explicitly that a symmetry of the Euler-Lagrange equations may not leave invariant the corresponding functional. It is shown that for the Kepler problem invariance under a two-parameter Lie group leads to a reduction of the order by four. Olver defines trivial conservation laws of the first and second kind and introduces the notion of characteristics of conservation laws. His comments on Noether's results are elucidating.

In the fifth chapter specialists as well as users of generalized symmetries will welcome Olver's detailed presentation on recursion operators, their construction (some "guesswork" is necessary) and uses. Olver shows that with generalized symmetries, Noether's theorem provides a one-to-one correspondence between variational symmetries and conservation laws through the use of adjoints of differential operators. Olver derives the Runge-Lenz vector.

The sixth chapter discusses the connections between symmetry groups, conservation laws, and reduction in order for systems in Hamiltonian form. In order to have coordinate-free results, Olver does not use special canonical coordinates. The fundamental object of study is the Poisson bracket. Olver shows that his results for ordinary differential equations in Lagrangian form result more naturally in a Hamiltonian framework. For systems of Euler-

Lagrange equations, first integrals arise from variational symmetry groups; for Hamiltonian systems the role of such groups is played by one-parameter Hamiltonian symmetry groups whose infinitesimal generators are Hamiltonian vector fields. Olver defines Hamiltonian symmetry groups in terms of a Poisson bracket.

In the final chapter on Hamiltonian methods for evolution equations, Olver shows that canonical coordinates do not generalize but the Poisson bracket does. He defines a Hamiltonian linear operator in terms of its Poisson bracket and gives the criterion for an operator to be Hamiltonian. It is shown how a partial differential equation can be put in Hamiltonian form. There is an important section on bi-Hamiltonian systems which are systems which can be written in Hamiltonian form in two distinct ways. An example is the Korteweg-de Vries equation. Bi-Hamiltonian systems are shown to have infinite sequences of generalized symmetries with corresponding explicit conservation laws, generated by a recursion operator. This operator is calculated from the two Poisson brackets so that such systems are "completely integrable". Some of the results in this chapter stem from crucial work of Arnol'd [**11**, **12**] and Gardner [**13**].

Olver discusses effectively most of the accomplishments in applying Lie groups to differential equations. He has unearthed important results unknown to many specialists. Olver does not discuss connections with separation of variables nor applications to boundary value problems and relegates his limited presentation on mappings mostly to the exercises. These omissions in no way reflect negatively on this exciting text.

Olver's textbook is a tour de force which will surely become a classic in the mathematical literature. I highly recommend it to both users and specialists.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 18, Number 1, January 1988

0273-0979/88 $1.00 + $.25 per page

The geometry of discrete groups, by Alan F. Beardon. Graduate Texts in Mathematics, vol. 91, Springer-Verlag, Berlin and New York, 1983, xii + 337 pp., $39.00. ISBN 0-387-90788-2

Let $\overline{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$ be the one-point compactification of $\mathbf{R}^n$, $n \geq 1$. The group G_n of Möbius transformations is the transformation group on $\overline{\mathbf{R}}^n$ generated by the translations

$$x \mapsto x + a, \qquad a \in \mathbf{R}^n, \tag{1}$$

and the inversion

$$x \mapsto x/|x|^2 \tag{2}$$

in the unit sphere. There are a number of reasons why Möbius transformations play a central role in the geometry of $\mathbf{R}^n$. For instance:

(a) According to a classical theorem of Liouville, if $n \geq 3$, every conformal map from one subregion of $\mathbf{R}^n$ to another is the restriction of a Möbius transformation.

(b) The sense-preserving transformations in G_2 are the fractional linear transformations

$$g(z) = (az + b)(cz + d)^{-1}, \qquad ad - bc = 1, \tag{3}$$

which are fundamental tools in geometric function theory.

(c) If we embed $\mathbf{R}^n$ in $\mathbf{R}^{n+1}$ in the usual way, by identifying $\mathbf{R}^n$ with $(e_{n+1})^{\perp}$, formulas (1) and (2) define an action of G_n on $\overline{\mathbf{R}}^{n+1}$. In fact G_n is the subgroup of G_{n+1} that maps the half-space

$$H^{n+1} = \{x \in \mathbf{R}^{n+1}; x \cdot e_{n+1} > 0\}$$

onto itself. H^{n+1} with the Poincaré metric $ds = |dx|/(x \cdot e_{n+1})$ is the $(n+1)$-dimensional hyperbolic space, and G_n is its isometry group.

(d) Every Riemannian manifold of constant negative curvature (-1) can be represented as the quotient of H^{n+1} by a discrete subgroup Γ of G_n. In particular, the classical uniformization theorem implies that almost all