RESEARCH ANNOUNCEMENTS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 18, Number 1, January 1988

CLASSICAL INVARIANT THEORY AND THE EQUIVALENCE PROBLEM FOR PARTICLE LAGRANGIANS

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ABSTRACT. The problem of equivalence of binary forms under the general linear group is shown to be a special case of the problem of equivalence of particle Lagrangians under the pseudogroup of transformations of both the independent and dependent variables. The latter problem has a complete solution based on the equivalence method of Cartan. This leads to the determination of a *universal function* which relates two particular rational covariants of any binary form. In essence, the main result is that two binary forms are equivalent if and only if their universal functions are identical. Extensions to forms in higher dimensions are indicated.

Consider a first-order variational problem

$$\mathcal{L}[u] = \int L(x,u,p)\,dx, \qquad x,u \in \mathbf{R},\ p \equiv \frac{du}{dx},$$

where the *Lagrangian* $L(x,u,p)$ is analytic on a domain $\Omega \subset \mathbf{R}^3$. Two Lagrangians L and $\tilde{L}$ are *equivalent* if there exists a change of variables $\tilde{x} = \varphi(x,u)$, $\tilde{u} = \psi(x,u)$ mapping one to the other. The change in the derivative is a linear fractional transformation

$$\tilde{p} = \frac{a\cdot p + b}{c\cdot p + d}, \tag{1}$$

where $a = \psi_u$, $b = \psi_x$, $c = \varphi_u$, $d = \varphi_x$. Equivalent Lagrangians must be related by

$$L(x,u,p) = (c\cdot p + d)\cdot \tilde{L}(\tilde{x},\tilde{u},\tilde{p}). \tag{2}$$

(This equivalence problem is a restricted version of the "true" Lagrangian equivalence problem, in which one can also add in a divergence term, solved in [7].)

Received by the editors March 26, 1987 and, in revised form, July 28, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 15A72, 49F05, 58A15.
Research supported in part by NSF grant DMS 86-02004.

Cartan [1] developed a powerful algorithm that will solve such equivalence problems, providing explicit necessary and sufficient conditions for equivalence. The first step is to reformulate the problem in terms of differential forms. Following Cartan, we introduce the "coframes"

$$\omega_1 = du - p\cdot dx, \qquad \omega_2 = L(x,u,p)\cdot dx, \qquad \omega_3 = dp,$$
$$\tilde\omega_1 = d\tilde u - \tilde p\cdot d\tilde x, \qquad \tilde\omega_2 = \tilde L(\tilde x,\tilde u,\tilde p)\cdot d\tilde x, \qquad \tilde\omega_3 = d\tilde p,$$

which are bases for the cotangent spaces $T^*\Omega$ and $T^*\tilde\Omega$ respectively. It is then easily seen that two Lagrangians L and $\tilde L$ are equivalent, if and only if there is a diffeomorphism $\Phi\colon \Omega \to \tilde\Omega$ such that the pull-back Φ^* transforms the coframes as follows:

$$\Phi^*(\tilde\omega_1) = A\cdot\omega_1, \qquad \Phi^*(\tilde\omega_2) = B\cdot\omega_1 + \omega_2, \qquad \Phi^*(\tilde\omega_3) = C\cdot\omega_1 + D\cdot\omega_2 + E\cdot\omega_3.$$

Here A,B,C,D,E are unspecified functions of x,u,p, with $A,E \neq 0$. This condition can be restated more symmetrically by introducing the "lifted" coframes

$$\theta_1 = A\cdot\omega_1, \qquad \theta_2 = B\cdot\omega_1 + \omega_2, \qquad \theta_3 = C\cdot\omega_1 + D\cdot\omega_2 + E\cdot\omega_3,$$
$$\tilde\theta_1 = \tilde A\cdot\tilde\omega_1, \qquad \tilde\theta_2 = \tilde B\cdot\tilde\omega_1 + \tilde\omega_2, \qquad \tilde\theta_3 = \tilde C\cdot\tilde\omega_1 + \tilde D\cdot\tilde\omega_2 + \tilde E\cdot\tilde\omega_3,$$

which live on $\Omega \times G$ and $\tilde\Omega \times G$, where G denotes the Lie group consisting of all matrices

$$\begin{bmatrix} A & 0 & 0 \\ B & 1 & 0 \\ C & D & E \end{bmatrix}, \qquad A,B,C,D,E \in \mathbf{R}, \ A,E \neq 0.$$

LEMMA 1 [1]. *Two Lagrangians L and $\tilde L$ are equivalent if and only if there is a diffeomorphism $\Psi\colon \Omega\times G \to \tilde\Omega \times G$ mapping the lifted coframes directly to each other:*

$$\Psi^*(\tilde\theta_1) = \theta_1, \qquad \Psi^*(\tilde\theta_2) = \theta_2, \qquad \Psi^*(\tilde\theta_3) = \theta_3.$$

The solution to this equivalence problem is now effected by the fundamental Cartan algorithm [**1,2,3**]. There are three distinct branches. If $L_{pp} \equiv 0$, then the variational problem is trivial. The other two branches depend on whether L_{pp} has the same or opposite sign to L in Ω. In both cases, the group reduces to the identity, whereby explicit formulas for the parameters A,B,C,D,E are determined, and the final structure equations take the form

$$d\theta_1 = -I\cdot\theta_1\wedge\theta_3 + \theta_2\wedge\theta_3, \qquad d\theta_2 = \pm\theta_1\wedge\theta_3, \qquad d\theta_3 = R\cdot\theta_1\wedge\theta_3 + S\cdot\theta_1\wedge\theta_3.$$

(The $\pm$ sign depends on the branch.) Thus, there are three fundamental invariants for the problem, denoted I, R and S. (However, the "generalized Bianchi identity" of [**2**] shows that S is a function of I and its derived invariants, so there are really only two independent invariants.) We find

$$I = \frac{L\cdot L_{ppp} + 3L_p\cdot L_{pp}}{|L|^{1/2}\cdot|L_{pp}|^{3/2}}, \tag{3}$$

while the explicit formulas for R and S are quite a bit more complicated [**2,7**]. However, if L depends only on the derivative p, then $R = S = 0$ and there is just one invariant.

Associated with the basic invariants I, R, S are their *derived invariants* I_j, R_j, S_j, $j = 1, 2, 3$, defined by the formula $dI = \sum I_j \cdot \theta_j$, etc. If L depends only on p, the only one of these which does not automatically vanish is

$$(4) \qquad I_3 \equiv J = \frac{|L|^{1/2}}{|L_{pp}|^{1/2}} \cdot \frac{\partial I}{\partial p}.$$

THEOREM 2. *For the Lagrangian equivalence problem, the derived invariants are functions of the fundamental invariants:*

$$I_j = F_j(I, R, S), \qquad R_j = G_j(I, R, S), \qquad S_j = H_j(I, R, S), \qquad j = 1, 2, 3.$$

Moreover, two nontrivial Lagrangians L and $\tilde{L}$ are equivalent if and only if

(a) *the ratios L_{pp}/L and $\tilde{L}_{\tilde{p}\tilde{p}}/\tilde{L}$ are either both positive or both negative,*

(b) *if an invariant for L is constant, the corresponding invariant for $\tilde{L}$ has the same constant value,*

(c) *the functions relating these invariants are identical: $\tilde{F}_j = F_j$, $\tilde{G}_j = G_j$, $\tilde{H}_j = H_j$, and*

(d) *the invariant equations*

$$I(x, u, p) = \tilde{I}(\tilde{x}, \tilde{u}, \tilde{p}), \qquad R(x, u, p) = \tilde{R}(\tilde{x}, \tilde{u}, \tilde{p}), \qquad S(x, u, p) = \tilde{S}(\tilde{x}, \tilde{u}, \tilde{p}),$$

have a common real solution.

In particular, if $L(p)$ does not explicitly depend on x or u, and is not an affine function of p, then there is just one nontrivial *universal function* F relating the invariants (3), (4): $J = F(I)$. The term "universal function" is used because, along with the sign, constant and solvability restrictions, it completely determines the equivalence class of a Lagrangian $L(p)$.

Turning to classical invariant theory, a homogeneous polynomial function

$$f(x, y) = \sum_{i=0}^{n} a_i x^i y^{n-i},$$

of two (real or complex) variables is called a *binary form* of degree n. Two forms f and $\tilde{f}$ are called *equivalent* if they can be transformed into each other by a suitable element of the real or complex general linear group GL(2). One of the principal goals of classical invariant theory [**5,6,8**] is the classification of binary forms using invariants or covariants. Despite the constructive methods used to generate the covariants themselves, it is by no means clear which of the many available covariants play the crucial role in the equivalence problem.

Given a binary form, let $g(p) = f(p, 1)$ be the associated nonhomogeneous polynomial. Note that the action of GL(2) reduces to the same linear fractional transformations (1), with corresponding action

$$g(p) = (c \cdot p + d)^n \cdot \tilde{g}(\tilde{p}) = (c \cdot p + d)^n \cdot \tilde{g}\left(\frac{a \cdot p + b}{c \cdot p + d}\right)$$

on the associated polynomials. Define the "Lagrangian"

$$L(p) = \sqrt[n]{g(p)}.$$

(For complex forms, L will be multiple-valued; for real forms, L will be defined and positive on the domain where g is positive. This leaves out negative semidefinite forms, which can be treated by replacing g by $-g$ *provided* it is kept in mind that for forms of even degree this does *not* correspond to a transformation in $\mathrm{GL}(2,\mathbf{R})$, and hence such forms are in a *separate* real equivalence branch.)

We now translate Theorem 2 into the language of classical invariant theory by expressing the invariants I and J in terms of known covariants. We first note that

$$L_{pp} = \frac{1}{n^2(n-1)} L^{1-2n} \cdot H, \tag{5}$$

where $H = f_{xx}f_{yy} - f_{xy}^2$ is the Hessian of f. (This formula gives an elementary proof of the classical result that f is the nth power of a linear form if and only if its Hessian vanishes identically; cf. [**8**, Proposition 5.3].) Assuming H is not zero, and $n > 2$, then

$$I = -\frac{(n-1)^{1/2}}{n^{3/2}} \frac{T}{|H|^{3/2}}, \qquad J = -\frac{n-1}{2n(n-2)} \frac{g \cdot U}{H^3},$$

where T and U are the Jacobian covariants

$$T = (f, H) = f_x \cdot H_y - f_y \cdot H_x, \qquad U = (H, T) = H_x \cdot T_y - H_y \cdot T_x,$$

re-expressed in terms of p. (For quadratic forms, I and J are both zero.) As a direct consequence of Theorem 2, we have the following fundamental theorem on the equivalence of binary forms (for simplicity, we slightly modify the covariants I and J).

THEOREM 3. *Let $f(x,y)$ be a binary form of degree n, which is not the nth power of a linear form. Define the absolute rational covariants*

$$I = \frac{T^2}{H^3}, \qquad J = \frac{f \cdot U}{H^3}. \tag{6}$$

*If I is not constant (so J does not vanish identically), define the **universal function** F so that $J = F(I)$. Then two binary forms f and $\tilde{f}$ are equivalent under the complex general linear group* $\mathrm{GL}(2,\mathbf{C})$ *if and only if either*

(a) *the covariants I and $\tilde{I}$ have the same constant values $I = \tilde{I}$, or*

(b) *I, $\tilde{I}$ are not constant, and the universal functions F and $\tilde{F}$ are identical: $F = \tilde{F}$.*

Therefore, the complete solution to the complex equivalence problem for binary forms depends on merely *two* absolute rational covariants—I and J! Actually, Theorem 2 provides a complete classification over the reals. Surprisingly, there are only *two* additional branch restrictions, depending in essence on the relative signs of f and H, along with the solvability of the real invariant equation $I(p) = \tilde{I}(\tilde{p})$ in the nonconstant case. (Note also that in the nonconstant case the equation $I(p) = \tilde{I}(\tilde{p})$ implicitly determines the linear fractional transformation taking f to $\tilde{f}$.) More rigorously, one needs to check whether f is positive semi-, negative semi- or in-definite, and whether H is positive semi-, negative semi- or in-definite on the domains where f is positive or negative. Some further consequences of the equivalence method follow.

THEOREM 4. *Let $f(x,y)$ be a binary form of degree n.*

(i) *If $H \equiv 0$, then f admits a two-parameter group of symmetries.*

(ii) *If $H \not\equiv 0$, and I is constant, then f admits a one-parameter group of symmetries.*

(iii) *If $H \not\equiv 0$, and I is not constant, then f admits at most a discrete symmetry group.*

COROLLARY 5. *A binary form f is complex-equivalent to a monomial, i.e. to $x^i \cdot y^{n-i}$, if and only if the covariant T^2 is a constant multiple of H^3.*

EXAMPLE.* For the binary quartic there are two important invariants, called i and j, cf. [**5**, §89]. To evaluate the universal function, we use the fundamental syzygy

$$T^2 = -\tfrac{1}{2}H^3 + \tfrac{1}{4}i \cdot f^2 \cdot H - \tfrac{1}{6}j \cdot f^3,$$

and the identity

$$U = \tfrac{1}{6}f \cdot (i \cdot H - j \cdot f),$$

cf. [**5**, pp. 98, 99]. Therefore the absolute covariants I and J have the form

$$I = -\tfrac{1}{2} + \tfrac{1}{4}i \cdot s^2 - \tfrac{1}{6}j \cdot s^3, \qquad J = \tfrac{1}{6}i \cdot s^2 - \tfrac{1}{6}j \cdot s^3, \qquad \text{where } s = f/H.$$

Thus, I is constant, and hence f is equivalent to a monomial, if either $i = j = 0$ or f is a constant multiple of H. (Actually, the first implies the second.) Otherwise, eliminating s, we see that the universal function $J = F(I)$ appears as the implicit solution to the cubic equation

$$6j^2(2I - 2J + 1)^3 = i^3(2I - 3J + 1)^2.$$

Using the classification of complex canonical forms for binary quartics given by Gurevich [**6**, p. 292], we see that cases I and II have nonconstant invariant I, while III and IV have constant invariant I. The classification over $\mathbf{R}$, cf. [**6**, Ex. 25.13,14], can also be effected using the relative signs of f and H; the details will appear elsewhere.

These methods are easily extended to forms in higher dimensions, e.g. ternary forms. The equivalence problem for homogeneous polynomials $f(\mathbf{z})$, $\mathbf{z} \in \mathbf{R}^k$, is easily translated into an equivalence problem for multiparticle Lagrangians $L(x, \mathbf{u}, \mathbf{p})$, $x \in \mathbf{R}$, $\mathbf{u} \in \mathbf{R}^{k-1}$, $\mathbf{p} = d\mathbf{u}/dx$, under the change of variables $\tilde{x} = \varphi(x, \mathbf{u})$, $\tilde{\mathbf{u}} = \psi(x, \mathbf{u})$. Indeed, we just let $\mathbf{p} = (p_1, \ldots, p_{k-1}) = (z_1/z_k, \ldots, z_{k-1}/z_k)$ be the corresponding homogeneous coordinates, and $L(\mathbf{p}) = \sqrt[n]{f(\mathbf{p}, 1)}$, where n is the degree of f. The equivalence problem is set up as before [**3**], and has been solved [**4**], although the solution has not yet appeared in the literature. The ternary case ($k = 3$) would be especially interesting to investigate in detail.

It is a pleasure to thank Niky Kamran and Bill Shadwick for igniting my interest in the Cartan equivalence method, and for many valuable discussions and correspondence on the subject. I would also like to thank Robert Gardner for helpful comments on an earlier version of this paper.

*In this example, the normalizations of H, T, U have been modified to conform with those in [**5**].

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