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SMOOTH NONTRIVIAL 4-DIMENSIONAL s-COBORDISMS

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ABSTRACT. This announcement exhibits smooth 4-dimensional manifold triads $(W; M_0, M_1)$ which are s-cobordisms, i.e. the inclusions $M_i \subseteq W$, $i = 0, 1$, are simple homotopy equivalences, but are not diffeomorphic or even homeomorphic to a product $M_i \times [0, 1]$.

The Barden-Mazur-Stallings s-cobordism theorem constitutes one of the foundational stones of modern topology. It asserts, in the smooth, piecewise-linear, or topological categories, that if W is a manifold of dimension at least six, with boundary components M_i, $i = 0, 1$, whose inclusions into W are simple homotopy equivalences, then W is necessarily a product (see [**K**, **H**, **RS**, **KS**]). For simply connected smooth manifolds of dimension at least six, this result had already been proven by Smale as the "h-cobordism theorem" [**Sm**], with the generalized Poincaré conjecture in higher dimensions as a corollary. The s-cobordism statement holds in dimensions one and two, and is equivalent to the Poincaré conjecture in dimension three. Freedman [**F1**, **F2**] proved the five-dimensional result for topological manifolds with fundamental group of polynomial growth (e.g. finite or polycyclic). Donaldson's extraordinary results imply the failure of the five-dimensional result in the smooth (or piecewise linear) category even for simply connected manifolds; by [**F1**] the resulting h-cobordisms will still be topological products. Using Freedman's results, the present authors produced some nontrivial orientable four-dimensional topological s-cobordisms [**CS1**, **CS2**]. (See [**MS**] for a nonorientable and definitely nonsmoothable example.) These topological constructions have been further studied and extended by Kwasik and Schultz [**KwS**].

We will now use a different construction to produce some nontrivial smooth s-cobordisms. Neither the construction nor the proof rely on any of the results cited above. Let M be a quaternionic space-form; i.e.

$$M = M_r = S^3/Q_r,$$

Q_r the quaternionic group of order 2^{r+2}. Then it is well known that the orientable manifold M has a one-sided Heegaard splitting

$$M = N(K) \cup H,$$

where $N(K)$ is the total space of an interval bundle over the Klein bottle K and H is a solid torus. Let E_0 be a closed tubular neighborhood of

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$K = K \times \{0\}$ in $M \times (-1,1)$. Then E_0 is a linear D^2-bundle over K with boundary the double of $N(K)$. Let

$$X = M \times [-1,1] - \text{Int } E_0.$$

The smooth s-cobordisms will be of the form

$$W = W_r = X \cup_{\partial E_0} E,$$

where E will be a locally trivial smooth fiber-bundle over K with fiber $T_0^2 = S^1 \times S^1 - \text{Int } D^2$, with $\partial E = \partial E_0$.

In fact, view $S^1 \subset \mathbf{C}$ and define ψ_i, $i = 1, 2$, by

$$\psi_1(x,y) = (y, yx) \quad \text{and} \quad \psi_2(x,y) = (y^{-1}, y^{-1}x^{-1}).$$

Note that $\psi_1^2 = \psi_2^2$. The Klein bottle K is the union of two Möbius bands, and it follows that there is a canonical T^2-bundle E_1 over K whose restrictions to the cores of the Möbius bands have monodromies ψ_1 and ψ_2 respectively. Since $\psi_i(1,1) = (1,1)$, this bundle has a cross-section, and there is a canonical way to identify a tubular neighborhood of its image with E_0. We then take $E = E_1 - \text{Int } E_0$ in the above definition of W. Clearly, W is an orientable smooth 4-manifold with two copies of $M = S^3/Q_r$ as boundary.

THEOREM. 1. *The smooth four-manifold W is an s-cobordism of M to itself.*

2. *W is not diffeomorphic or even homeomorphic to a product $M \times [-1,1]$.*

It can also be shown that W is not homeomorphic to any of the topological s-cobordisms of [**CS2**], and the smoothability of any of them remains open.

The proof of 1 uses Van Kampen's theorem and other well-known arguments in homotopy and simple homotopy theory. However, note that the restriction of a suitable diffeomorphism of T^2 isotopic to ψ_i represents a square-root of the monodromy of the figure-eight knot.

We indicate the proof of 2 for the case $r = 1$, the quaternion group of order eight. Let P be obtained from W by identifying $M \times \{-1\}$ with $M \times \{1\}$. Then we explicitly construct a framed 5-manifold U with the following properties:

1. $\partial U = P$.

2. There is a retraction $r\colon U \to M$ inducing isomorphisms on fundamental groups and homology with $\mathbf{Z}_2$ coefficients.

3. If U_4 and U_8 are the 4-fold and 8-fold covers of U, respectively, then $|H_2(U_8)||H_2(U_4)|^{-1} \equiv \pm 7 \pmod{16}$.

By contrast, we show that were W a product and U as above satisfying 1 and 2, the quotient (of odd integers) in 3 would necessarily be congruent to $\pm 1 \pmod{16}$. Because of the possible choices for P and r, the proof is somewhat involved. It uses the fact, due independently to J. H. Rubinstein [**R**] and the present authors, that a diffeomorphism or homeomorphism of M homotopic to the identity will necessarily be isotopic to it. In the course of the proof, the remaining ambiguity of [**KwS**] concerning the classification of topological s-cobordisms of M to itself is resolved, and a remark in [**CS2**] is corrected.

It would be interesting to know if the universal covering space of W is diffeomorphic to $S^3 \times [0,1]$. This is similar to the situation for the exotic $\mathbf{RP}^4$ of **[CS3]**, whose covering space is also potentially exotic **[AK]**. It is also of interest to observe that for the case $r = 1$, W can be embedded as a codimension zero submanifold of a smooth homotopy 4-sphere.

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