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INDUCTION THEOREMS FOR INFINITE GROUPS

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The purpose of this paper is to announce Theorems 1 and 4 below. These may be viewed as generalizations of theorems of Brauer, Swan, and Artin [**11**] to certain classes of infinite groups.

THEOREM 1. *Let G be a virtually polycyclic group. Let U be a G-graded ring with a unit in each degree, such that U_1 is Noetherian. Then the induction map*

$$\bigoplus_{\substack{H\subset G\\ \text{finite}}} K_0'U_H \to K_0'U_G \tag{1}$$

is surjective, where U_H is the part of U supported on H, for each $H \subset G$.

The proof depends on a structure theorem for such U.

Added in proof: I wish to thank Hyman Bass for carefully planning my course of graduate study to bring me into contact with this constellation of research questions. The idea for the structure theorem comes from a case of the general unpublished conjecture of Farrell and Hsiang that leads to recent work of Farrell-Jones and F. Quinn. The conjecture about K_0' was independently posed by S. Rosset [**7**] based on ring-theoretic evidence. I wish to thank Tom Farrell for showing me his conjecture with Hsiang and expressing confidence in the approach to Theorem 1.

In the special case that k is a field of characteristic 0 and U is the group algebra kG with the natural grading, Theorem 1 is equivalent to the following result announced by F. Quinn [**8**] (at least for $k = \mathbf{Q}$) in establishing Farrell and Hsiang's conjecture [**4**]:

$$K_0(kG) \simeq \varinjlim_{H\in\mathcal{F}(G)} K_0(kH) \tag{2}$$

($\mathcal{F}$ = Frobenius category of finite subgroups). In effect, as kG and all kH are regular, we may identify K_0 and K_0' via the Cartan map, so (2) $\Rightarrow$ (1). Conversely, in the diagram

$$\begin{array}{ccc} \varinjlim_{H\in\mathcal{F}(G)} K_0(kH) & \to & K_0(kG) \\ \downarrow r & & \downarrow r \\ \varinjlim_{H\in\mathcal{F}(G)} T(kH) & \hookrightarrow & T(kG) \end{array}$$

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where the vertical maps are Bass's rank map [**1**], once the top arrow is surjective it is automatically an isomorphism, so (1) $\Rightarrow$ (2).

In case k is instead a Noetherian Hilbert ring and $U = kG$ for G crystallographic, K. A. Brown, J. Howie, and M. Lorenz have already shown that $(1) \otimes \mathbf{Q}$ is surjective [**3**]. They made use of Roseblade's structure theory of group algebras of poly $\mathbf{Z}$ by finite groups.

Rosset showed [**7**] when k is a field and G is a prime virtually polycyclic group, letting $Q_c(kG)$ denote the simple Artinian classical fraction ring of kG, and writing

$$a = \operatorname*{lcm}_{\substack{H\subset G\\ \text{finite}}} |H|,$$

$$b = \text{length}(Q_c kG),$$

$$c = \text{least common denominator } \chi(M), \ M \text{ f.g. } kG\text{-module},$$

that $a|b$ and $b|c$. Also he showed that whenever the induction map

$$\bigoplus_{\substack{H\subset G\\ \text{finite}}} K_0'(kH) \to K_0'(kG)$$

is surjective, $c|a$.

However, Theorem 1 implies that this map is indeed surjective, so we obtain the solution of the Goldie rank conjecture:

THEOREM 2 (Rosset, Goldie rank conjecture).

$$\text{length } Q_c kG = \operatorname*{lcm}_{\substack{H\subset G\\ \textit{finite}}} |H|.$$

In a joint paper with P. Kropholler and P. Linnell, we plan to apply Theorem 1 to extend the statement of Theorem 2 to a larger class of groups. In particular, we will deduce the Zero-Divisor Conjecture for solvable groups.

To illustrate the technique,

THEOREM 3. *Let U be a G-graded ring with a unit in each degree. Suppose that G is virtually polycyclic and for each $H \subset G$ finite, U_H is an Ore domain. Then U is an Ore domain.*

Letting $S = U_1 - 0$, each $S^{-1}U_H$ is a division ring when $\widetilde{K_0'}(S^{-1}U_H) = 0$. Then by Theorem 1, $\widetilde{K_0'}(S^{-1}U) = 0$, so that Noetherian ring is necessarily an Ore domain; so U must have been.

Note that Theorem 3 is not weakened by replacing the words "virtually polycyclic" by "finitely generated virtually abelian," because once the theorem holds for all finite extensions of the factors of a finite composition series of a group Γ, it holds as well for Γ. Note also that it suffices to verify the conclusion for finitely generated subgroups; so, for instance, "finitely generated virtually abelian" can be replaced by "virtually abelian." For full details in the general case refer to our joint paper mentioned above.

THEOREM 4. *Suppose $G \subset \mathrm{GL}_m(F)$ for some field F, $m \geq 1$; k is a normal domain of characteristic 0. Let $r\colon G \to k$ be a virtual rank function. Then there is a number $e \geq 1$ and finite hyperelementary subgroups $H_1, \ldots, H_n$ of G such that for all $g \in G$*

$$e \cdot r(g) = \sum_{i=1}^{n} \frac{1}{|H_i|} \sum_{\substack{h \sim g \\ \text{in } G}} \chi_i(h^{-1}) \tag{3}$$

where for each i, $\chi_i\colon H_i \to k$ is a virtual character of finitely generated projective kH_i-modules. Take e as small as possible.

For any group G, any $k \subset \mathbf{Q}$, and any virtual rank function $f\colon G \to k$ [**1**], Linnell showed [**5**], when $r(g) \neq 0$, that

$$r(g) = r(g^p), \qquad p \notin k^\times \text{ prime.} \tag{4}$$

We show that for $k \subset \mathbf{C}$, any group G and any virtual rank function $f\colon G \to k$, the finite abelian extension $K = \mathbf{Q}(\{r(g)\colon g \in G\})/\mathbf{Q}$ (see [**2**]) satisfies

$$\begin{aligned} &1.\ \ d(K/\mathbf{Q}) \in k^\times, \\ &2.\ \ r(g^p) = (K/\mathbf{Q}, p)(r(g)) \quad \text{for } p \notin k^\times, \end{aligned} \tag{5}$$

where $(K/\mathbf{Q}, p)$ is the Artin symbol and $d(K/\mathbf{Q})$ is the discriminant. This represents an improvement of Bass's [**1**], which asserted the second part for almost all unramified $p \notin k^\times$. Both parts were suggested by K. S. Brown (private correspondence) before Linnell's preprint became available.

Theorem 4 is a consequence of (5) and elementary techniques of [**9**].

REMARK 5. *If k is a field of characteristic 0 and G is virtually polycyclic,* (2) *implies that $e = 1$ and that, given finitely generated projective kG-modules P and P' with rank functions r and r',*

$$r = r' \Leftrightarrow P \sim P' \text{ stably isomorphic}$$

so $K_0(kG)$ is isomorphic to the group of functions $G \to k$ of the form (3).

The hypothesis that G is a linear group is not necessary in Theorem 4. In fact Theorem 4 goes through, without the linearity assumption, for Formanek's groups of type (D) (cf. [**1**]).

Also, using Theorem 4 and Kaplansky's theorem [**6**],

COROLLARY 6. *Let $P(T) \in k[T]$ be monic of discriminant Δ, k a domain of characteristic* 0 *and G a group of type* (D). *If the order of no finite subgroup of G is invertible in k, and $\Delta \in k^\times$, then all solutions in kG of the equation*

$$P(T) = 0$$

actually lie in k.

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