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AMALGAMATIONS AND THE KERVAIRE PROBLEM

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ABSTRACT. Following S. Brick, a 2-complex X is called "Kervaire" if all systems of equations, with coefficients in arbitrary groups G and the attaching maps of X as the words in the variable letters, are solvable in an overgroup of G. An obstruction theory is developed for solving equations modeled on $Z = X\amalg_\Gamma Y$, where X and Y are Kervaire 2-complexes and Γ is a subgraph of $Z^{(1)}$, each connected component of which injects at the π_1-level into $\pi_1(Z)$. A 2-complex of the form $K\langle\vec{x},\vec{y}|\ w(\vec{x}) = w'(\vec{y})\rangle$ is Kervaire, where $w(\vec{x})$ and $w'(\vec{y})$ are (not necessarily reduced) words which do not freely reduce to 1.

The Kervaire problem [**7**, p. 403] originally asked whether a nontrivial group can be killed by adjoining a single free generator and a single relator. This problem has been vastly generalized by Howie [**5**], who asked whether a system of equations over an arbitrary coefficient group G, whose words in the variable letters are the attaching maps of a 2-complex X with $H_2(X) = 0$, is solvable in an overgroup of G. It is convenient to introduce a terminology due to S. Brick [**1**] who calls a 2-complex X *Kervaire* iff all systems of equations over all coefficient groups G modeled on the attaching maps of X are solvable in an overgroup of G. Thus, e.g., the dunce hat $K\langle x|xx\bar{x}\rangle$ is Kervaire because Howie has shown that the equation $axbxc\bar{x} = 1$, with $a, b, c \in G$, can always be solved in an overgroup of G [**6**].

In this terminology, a nontrivial group can never be killed by adjoining a single free generator and a single relator iff the 2-complex $K\langle x|w(x)\rangle$ is Kervaire, where $w(x)$ is a word in x and x^{-1} whose exponent sum in x is ± 1.

For a 2-complex with one 2-cell $X = K\langle x_1, x_2, \ldots, x_n|w(\vec{x})\rangle$ Howie's problem can be shown (nontrivially) to imply that X is Kervaire iff $w(\vec{x})$ does not freely reduce to 1 (the "if" assertion is the nontrivial one here). Since $X = K\langle\vec{x}|w(\vec{x})\rangle$ can be easily shown to be Cockcroft iff $w(\vec{x})$ does not freely reduce to 1, Howie's problem for 2-complexes X with one 2-cell amounts to

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the assertion that X is Kervaire iff X is Cockcroft (recall a 2-complex X is Cockcroft iff the Hurewicz homomorphism $\pi_2(X) \to H_2(X)$ is zero).

We can prove

THEOREM 1. *Let $x_1^\pm, x_2^\pm, \ldots, x_n^\pm$ and $y_1^\pm, \ldots, y_m^\pm$ be disjoint alphabets and let $w(\vec{x})$ and $w'(\vec{y})$ be words in these alphabets respectively which do not freely reduce to 1. Then $K\langle x_1, \ldots, x_n, y_1, \ldots, y_m | w(\vec{x}) = w'(\vec{y})\rangle$ is Kervaire.*

This result can be stated in the equivalent form below, more appealing to topologists, by recalling the connected sum $X \# Y$ of two 2-complexes [**8**]. One chooses imbeddings of the disc D^2 in X and Y respectively, each with one point contact with $X^{(1)}$ and $Y^{(1)}$, one bores out the interiors of the discs, and one identifies their boundaries to get $X \# Y$. The construction depends sensitively on the choice of imbeddings of discs.

THEOREM 2. *Let X and Y be Cockcroft 2-complexes each possessing only one 2-cell. Then $X \# Y$ is Kervaire (for all choices of imbedded discs in X and Y as above).*

The main technical innovation is an obstruction theory for deciding when $Z = X_1 \coprod_\Gamma X_2$ is Kervaire provided Γ is a subgraph of $Z^{(1)}$ such that π_1 of each connected component of Γ injects into $\pi_1(Z)$ (S. Brick calls such an inclusion $\Gamma \to Z$ *π_1-injective* [**1**]). Let $f\colon (D^2, S^1) \to (X, \Gamma)$ be a combinatorial map (for some cell structure on D^2). We define the obstruction element $\Lambda(f) \in G_f * \langle E(\Gamma)\rangle$ to be the product in order of corner labels and edge labels in one full circuit around ∂D^2; here G_f is the factor group of the corner group [**4**] of X modulo interior vertex labels of f and $\langle E(r)\rangle$ denotes a free group freely generated by an oriented set of edges of Γ. The technical result is the following

THEOREM 3. *Let $Z = X_1 \coprod_\Gamma X_2$, where the inclusion $\Gamma \to Z$ is π_1-injective. Assume that X_1 and X_2 are Kervaire and that all obstruction elements $\Lambda(f) = 1$ for all maps $(D^2, S^1) \xrightarrow{f} (X_i, \Gamma)$, $i = 1, 2$. Then Z is Kervaire.*

An example where all obstructions $\Lambda(f)$ vanish is where Γ is 2-sided in Z.* In this case Theorem 3 implies as a corollary a result of Brick's thesis [**1**]: if Γ is a subgraph of $Z^{(1)}$ such that the inclusion $\Gamma \to Z$ is π_1-injective and Γ is 2-sided in Z and if in addition the result of cutting Z along Γ is Kervaire, then Z is Kervaire.

To apply Theorem 3 we need to calculate obstructions. Let $X = K\langle x_1, \ldots, x_n, t | t = w(\vec{x})\rangle$ and let $\Gamma = K\langle t | \ \rangle$, a subgraph of $X^{(1)}$. The inclusion $\Gamma \to X$ is π_1-injective iff the word $w(\vec{x}) \in F(\vec{x})$ does not freely reduce to 1. We prove

THEOREM 4. *For any combinatorial map $f\colon (D^2, S^1) \to (X, \Gamma)$, where X and Γ are as defined immediately above and where $w(\vec{x})$ does not freely reduce to 1, one has $\Lambda(f) = 1$.*

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*Γ is called "2-sided" in Z if it is *bicollared:* so Γ is identified with $\Gamma \times \{1/2\}$ where $\Gamma \times [0, 1]$ is a product neighborhood of Γ in Z.

The proof of Theorem 4 proceeds by assuming f is reduced (so no two 2-cells of D^2 with an edge e in common are mapped mirror-wise across e) and showing that, by small cancellation type arguments, in this reduced case the domain has a vertex of valence 1 in its 1-skeleton. This enables us to do 2-bridge moves and at the same time reduce the size of $w(x)$ by cancelling an adjacent pair of cancelling letters. The argument proceeds by an induction on the length of $w(\vec{x})$, the induction beginning when w is a reduced word ($\neq 1$); in this case one sees directly no such reduced maps f can exist.

Theorem 2 follows from Theorems 3 and 4 by appealing to the subdivision theorem for Kervaire complexes [**1**] and by observing that the complex X in Theorem 4 collapses onto a graph and is hence Kervaire.

Similar arguments establish the following result. Recall that a 2-complex X is called diagrammatically reducible (DR) [**4**] if there are no reduced combinatorial maps of S^2 to X.

THEOREM 5. *Let $w_i(\vec{x}), i \in I$, be a set of words in the alphabet $\vec{x} = (x_1^{\pm}, \ldots, x_n^{\pm})$ and assume that the elements in the free group $F(\vec{x})$ these words $w_i(\vec{x})$ represent freely generate the subgroup S of $F(\vec{x})$. If no proper initial segment of any word $w_i(\vec{x})$ represents an element of S, then the 2-complex*

$$K\langle x_1, \ldots, x_n, y_1, \ldots, y_n | w_i(\vec{x}) = w_i(\vec{y}), i \in I\rangle$$

is diagrammatically reducible.

COROLLARY. *If F is a free group and $A \leq F$, then the double of F along A, $F *_A F$, has a DR presentation.*

It is an open question whether every aspherical 2-complex is homotopy equivalent to a DR 2-complex (see [**2**, §6] for additional examples, drawn from 3-manifold theory, where this is true).

Theorem 5 above has an amusing illustration. It follows immediately that the presentation $\langle x, y, z, w | x^n y^n z^n w^n, \ \forall n > 1\rangle$ is DR. This implies [**4**] that for any group G and sequence of elements $a_n \in G$, $n \geq 1$, the system of equations

$$a_n = x^n y^n z^n w^n, \quad \forall n \geq 1,$$

can be simultaneously solved in an overgroup of G.

Another explicit calculation of the obstruction element $\Lambda(f)$ shows there is a 2-complex which is Cockcroft but not Kervaire. Explicitly we have

THEOREM 6. *Let $X = K\langle x, y, t | x^2, y^2, t = xy\rangle$. Let $\Gamma = K\langle t| \ \rangle$, a π_1-injective subgraph of X. Then the double Z of X along Γ, $Z = X \amalg_{\Gamma} X$, is Cockcroft and diagrammatically aspherical but not Kervaire.*

"Diagrammatically aspherical" here means that given any combinatorial map of a cell structure S^2 to Z, some sequence of diamond moves exists which splits off a component 2-sphere with precisely two faces. The example Z of Theorem 6 is interesting because the homotopy equivalent 2-complex

$$W = (X \times (0))_{\Gamma \times (0)} (\Gamma \times I)_{\Gamma \times (1)} (X \times (1))$$

is Kervaire, as one sees by applying Brick's 2-sided π_1-injective theorem quoted after Theorem 3. It follows that the property of being Kervaire is not a homotopy type invariant of 2-complexes.

Suppose now that $X = K(P)$, where P is the finite presentation $P = \langle x_1, x_2, \ldots, x_n, t_i \ (i \in I) | t_i = w_i(\vec{x}), \ i \in I\rangle$, and let $\Gamma = K\langle t_i(i \in I) | \ \rangle$, a subgraph of $X^{(1)}$ (so X collapses cellularly onto a subgraph of $X^{(1)}$ with $E(\Gamma)$ as the set of free edges for the collapse). Let $Z = X \coprod_{\Gamma} X$, the double of X along Γ. It is easy to see that the inclusion $\Gamma \to X$ is π_1-injective iff Z is Cockcroft iff Z is aspherical iff $\{w_i(\vec{x}), \ i \in I\}$ is freely independent in $F(\vec{x})$.

THEOREM 7. *If Z is Kervaire, then the inclusion $\Gamma \to X$ is π_1-injective. Furthermore if $\Gamma \to X$ is π_1-injective and we assume either a positive solution to Howie's problem or the invariance of Kervaire complexes (with one vertex) under Andrews-Curtis moves, then Z is Kervaire.*

Theorem 5 is used in proving the last assertion in Theorem 7 as follows. If $\{w_i(\vec{x}), i \in I\}$ is independent, then one may do Nielsen moves to transform this collection to a Schreier basis for the subgroup generated; here Theorem 5 applies. On the other hand Nielsen moves on $\{w_i(\vec{x}), i \in I\}$ correspond to Andrews-Curtis moves on Z, so invariance of the Kervaire property under these latter moves implies that Z is Kervaire.

In this connection I have developed an algorithm for generating all reduced disc diagrams $f\colon (D^2, S^1) \to (X, \Gamma)$ with (X, Γ) as in Theorem 7. The algorithm is "smart" in the sense that it can select certain diagrams for which $\Lambda(f) = 1$ because of the known positive results about the Howie problem. Hand computations have so far led to no "interesting" diagrams, where a diagram is called "interesting" if these selection rules don't automatically imply $\Lambda(f) = 1$. The algorithm ought to be programmed on a high-speed computer, to continue the search for "interesting" diagrams.

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