essential, therefore, to the understanding of nonlinear phenomena as well as the operator to study in the first model problems.

The early results were summarized fifteen years ago by Berger, Gauduchon, and Mazets in *Le spectre d'une variété riemannienne*,[1] which at that time well represented the literature on the "geometry of the Laplace operator." Growing out of a set of lectures from the late sixties, the purpose of these notes was to acquaint graduate students or newcomers to the field of eigenvalues on manifolds with the basic background and results. One of the high points was the asymptotic expansion for the heat kernel which enabled one to read off geometric invariants of the manifold such as the volume and integral of scalar curvature from the "high-end" behavior of the spectrum. There were also a few scattered results on the first nonzero eigenvalue estimates. However, with the currently vigorous activity, it is high time for another overview of newer results and, perhaps more importantly, of the techniques from the field. To this end, Isaac Chavel offers his *Eigenvalues in Riemannian geometry*.

From his very particular vantage point, Chavel gives his summation of what is known about the interplay of the spectrum of the Laplacian and geometry in the book *Eigenvalues in Riemannian geometry*. The principal focus of this book is on the relationship between the lower spectrum (as opposed to asymptotics of the spectrum) and the geometry of the manifold. A substantial portion of this theory was developed in the last two decades and hence Chavel's book can be viewed as an organized (and desperately needed) update on the field. As well as being a modern and extensive list of theorems about eigenvalues, the book also covers sufficient (yet minimal) background material in geometry and elliptic PDEs so that it can be used as a graduate text.

The book serves as an excellent reference for areas and techniques which the author favors. However, beginners may find the presentation directionless and consisting of an assembly of isolated theorems. It would have been beneficial to the readers if more geometric results via eigenvalues were discussed. Undoubtedly, efforts to present another viewpoint on the subject will be made.

PETER LI

[1] An updated bibliography has been compiled by P. Bérard and M. Berger, and published as an appendix in *Spectral geometry: Direct and inverse problems* by P. Bérard.

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 16, Number 2, April 1987

0273-0979/87 $1.00 + $.25 per page

Equilibrium capillary surfaces, by Robert Finn, Grundlehren der mathematischen Wissenschaften, vol. 284, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1986, xi + 245 pp., $57.00. ISBN 0-387-96174-7

Robert Finn in the preface to his book writes:

"Capillarity phenomena are all about us; anyone who has seen a drop of dew on a plant leaf or the spray from a waterfall has observed them. Apart from their frequently remarked poetic qualities, phenomena of this sort are so familiar as to escape special notice. In this sense the rise of liquid in a narrow tube is a more dramatic event that demands and at first defied explanation; recorded observations of this and similar occurrences can be traced back to times of antiquity, and for lack of explanation came to be described by words deriving from the Latin word 'capillus', meaning hair."

In fact the first written description of the observation of the capillarity phenomena in narrow tubes was made by the Italian scientist Nicolò Aggiunti (1600–1635) in a booklet, never published, by the title "Un libro di problemi vari geometrici e di speculazioni, ed esperienze fisiche" [1].

Aggiunti wrote in his little book that "lo scoprimento del moto occulto dell'acqua risolverà moltissimi problemi" [the discovery of the hidden motion of water will solve many problems]. Among them "Perché l'acqua non si livelli in un vaso così fatto [Figure 1], ma sia più alta nella cannella angusta" [why the water is not at the same level in a vase like this, but is higher in the narrow tube].

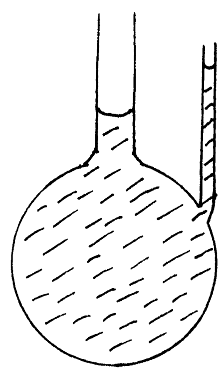

FIGURE 1

It is a commonplace that the first attempts to explain observed phenomena connected with capillarity go back to Leonardo da Vinci. "Enfin, deux observations capitales, celle de l'action capillaire et celle de la diffraction, dont jusqu'à présent on avait méconnu le véritable auteur, sont dues également à ce brillant génie" [2].

Some effects of capillary action were also known to the Muslim natural philosophers of the 12th century, such as Al-Khazini [3, 4, 5].

The modern history of capillarity attraction starts with Young [6] and Laplace [7] at the beginning of the 19th century. Laplace was able to show that the mean curvature H of the free surface of a liquid inside a narrow tube is proportional to the pressure change across the surface; this capillary pressure is due to the presence of molecular forces that have an extremely short range of action. This was Segner's 'tenacitas' [8], a first vague idea of what we today call surface tension; the idea of Segner was not completely correct, as was pointed out by Bikerman [9].

The notion of mean curvature of a surface was introduced by T. Young (1805) and P. S. Laplace (1806) just for characterizing quantitatively the rise of liquid in a narrow tube. The Laplace or Young-Laplace equation can be written as

$$P = \sigma(1/R_1 + 1/R_2)$$

where P is the pressure, σ is the surface tension, R_1 and R_2 are the two principal radii of curvature; so for the height u of the surface above the level corresponding to atmospheric pressure we have

$$\tfrac{1}{2}ku = H(1/R_1 + 1/R_2)$$

where k is a physical constant.

All these go back to the beginning of the 19th century. Why then still work on the problem? A first observation is that if we assume that the surface can be described as a graph of a function

$$z = u(x, y)$$

then we have for the mean curvature H

$$2H = au_{xx} + 2bu_{xy} + cu_{yy},$$

where

$$a = \frac{1 + u_y^2}{\left(1 + |Du|^2\right)^{3/2}}, \qquad b = \frac{-u_x u_y}{\left(1 + |Du|^2\right)^{3/2}}, \qquad c = \frac{1 + u_x^2}{\left(1 + |Du|^2\right)^{3/2}},$$

and so we obtain

$$au_{xx} + 2bu_{xy} + cu_{yy} = ku \tag{1}$$

for the height u of the surface.

The P.D.E. (1) is nonlinear, elliptic but not uniformly elliptic. Thomas Young in his "Essay" introduced also the idea that in equilibrium configuration the fluid meets the bounding walls in a constant angle γ depending only on the materials (Figure 2).

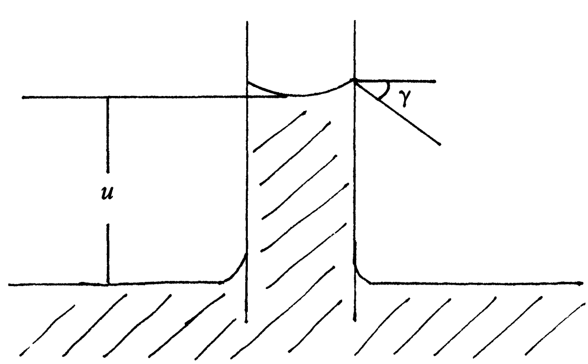

FIGURE 2

In 1830 Gauss proposed a new method to treat the problem. Gauss based his reasoning on the principle of virtual work, according to which the energy of a mechanical system in equilibrium is unvaried under arbitrary virtual displacements consistent with the constraints [**10**].

For a system of fluid and gas and rigid bounding walls the energy in question can be divided into four terms:

(1) Free surface energy: this energy must be proportional to the surface area A and so

$$E_s = \sigma A,$$

where σ is the surface tension.

(2) Wetting energy: this is the adhesion energy between fluid and the walls,

$$E_w = -\sigma\beta A^*,$$

where $\beta > 0$ in a 'wetting' configuration (Figure 2) and is called the relative adhesion coefficient between the fluid and the walls; A^* is the area wetted by the fluid.

(3) Gravitational energy:

$$E_g = \int \Lambda\rho\, dx$$

where ρ is the local density and Λ a potential energy for unit mass, depending on the position within the media.

(4) Volume constraint: in many problems the constancy of volume of fluid is a constraint that must be respected. We can write

$$E_V = \sigma\lambda V$$

where V is the volume and λ a Lagrange multiplier to be determined.

So the total energy is given by

$$E = \sigma\left\{A - \beta A^* + \frac{1}{\sigma}\int \Lambda\rho\, dx + \lambda V\right\}.$$

In order to apply the principle of virtual work we introduce a virtual displacement. If we consider a capillary surface sufficiently small that it can be represented as a graph $z = u(x, y)$ over a domain Ω, we then consider any variations with their support in Ω. Consider also the case of constant ρ and $\Lambda = gu$, the gravitational potential. Then we can write for the total energy, restricting only to terms that will be varied,

$$E = \sigma\left\{\int_\Omega \sqrt{1 + u_x^2 + u_y^2}\, dw + \frac{g\rho}{\sigma}\int udw + \lambda\int_\Omega u\, dw\right\}.$$

Using

$$\tilde{u}(x, y; \varepsilon) = u(x, y) + \varepsilon\eta(x, y),$$

as a virtual displacement for a candidate $u(x, y)$ for an equilibrium surface, with a smooth η and $\operatorname{supp}\eta \subset \Omega$, we obtain

$$\operatorname{div}\mathbf{T}u = \kappa u + \lambda; \qquad \mathbf{T}u = \frac{1}{\sqrt{1 + |\nabla u|^2}}\nabla u.$$

We can also put $\lambda = 0$ in the case of the configuration of Figure 2, in which the tube has infinite length in a large container. We then obtain

$$\operatorname{div} \mathbf{T}u = \kappa u$$

with

$$\kappa = \rho g/\sigma$$

called the capillary constant. Moreover if $\Sigma = \partial\Omega$ and ν is the exterior normal to Σ, and γ the angle between the surface S and the cylinder wall we have

$$\gamma \cdot \mathbf{T}u = \beta = \cos\gamma,$$

thus determining the constant angle γ.

We can now make the fundamental observation that explains one of the reasons for studying capillary problems: *the problem of finding a capillary surface is a purely geometric one, that is, to find a surface whose mean curvature is a prescribed function of position and which meets prescribed rigid boundary walls in a prescribed angle.*

$$\begin{cases} \operatorname{div} \mathbf{T}u = \kappa u & \text{in } \Omega, \quad (2) \\ \gamma \cdot \mathbf{T}u = \cos\gamma & \text{in } \Sigma. \quad (3) \end{cases}$$

Now a large number of the modern results on capillary surfaces are devoted to establishing the existence of solutions for the problem (2), (3). The first general result was obtained only in 1973 using the variational approach [**11**]. As we have seen, the energy functional consists of a 'surface integral' plus a 'volume integral'. Now the problem is that the classical definition of surface area is rather inadequate for treating this type of problem. A satisfactory theory of surface area for a general class of surfaces of codimension one in $\mathbf{R}^n$, $n \geqslant 2$, has been developed by E. De Giorgi in the fifties, and then by M. Miranda, M. Giaquinta, E. Giusti, and others [**12–15**]. Independently the ideas of geometric measure theory were developed by H. Federer, W. H. Fleming, F. J. Almgren, W. K. Allard, and others, and have been used effectively by Jean Taylor to consider boundary regularity for capillarity problems [**16–20**].

The idea is to look for the solution in a class of competing functions sufficiently large that compactness of a minimizing sequence can be guaranteed; the variational condition then leads to the uniqueness and regularity of the limit function. The class of functions considered by De Giorgi's method is the class of BV-functions, functions of bounded variation on Ω. As Finn pointed out, in his chapter devoted to existence theorems,

> Such an approach might seem at first glance to be hazardous for the present problems, since the boundary condition (3) involves derivatives on the boundary, where differentiability of weak solutions is usually difficult to prove. In fact, the minimizing function produced by the variational procedure is known initially only to have a generalized L^1 trace on the boundary, so that (3) can be defined only in a very weak sense. Nevertheless, the minimizing

property suffices for showing the uniqueness of the limit function, and for identifying it with the smooth solution whenever a smooth solution exists.

So the energy functional for the capillary surfaces in cylindrical vertical tubes can be expressed as

$$\int_{\Omega} \sqrt{1 + |Du|^2} - \beta \int_{\partial\Omega} u\, dH_{n-1} + \frac{\kappa}{2} \int_{\Omega} u^2\, dw$$

where Ω, open and bounded in $\mathbf{R}^n$, has Lipschitz boundary, and $u \in \mathrm{BV}(\Omega)$.

An important step for the existence result is to obtain a general result regarding the trace of BV-functions,

$$\int_{\partial\Omega} u\, dH_{n-1} \leqslant \alpha \int_{\Omega} |Du| + \delta \int_{\Omega} u\, dw; \qquad \delta > 0,\ \alpha \geqslant 1.$$

It implies that capillary surfaces always exist for β in the range

$$0 < \beta < 1/\sqrt{1 + L^2} \quad \text{(wetting case)},$$

where L is the Lipschitz constant of $\partial\Omega$. In fact no solutions with bounded energy can exist, for example, for a domain in the form of a circular sector where θ is the angle of the sector when $\beta > 1/\sqrt{1 + L^2}$, that is, when $\theta + 2\gamma < \pi$. The discontinuity at $\theta + 2\gamma = \pi$ is also confirmed by physical experiments **[21]**.

I have only considered the case of the graph of a function $u = u(x, y)$ in Ω to give an idea of what type of problems occur. But many other questions can be considered: the case of drops, pendent and sessile, the parametric capillarity problems, boundary regularity, the symmetric case for vertical tubes, estimates for solutions, uniqueness. Many of these problems have been considered in the last years using either variational techniques or other methods. More recently the case of rotating drops has also been treated using a variational approach **[22, 23]**.

Finn's book is an exhaustive and clear description of all the more recent results, excluding rotating drops. It is not a simple collection of the latest works in this area but a successful attempt to give a unified and comprehensible treatment of all the problems connected with capillary phenomena.

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