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Scattering by obstacles, by A. G. Ramm, D. Reidel Publishing Company, Dordrecht, Boston, Lancaster, Tokyo, 1986, xiv + 423 pp., $89.00. ISBN 90-277-2103-3.

For well over a hundred years, scattering theory has played a central role in mathematical physics. From Rayleigh's explanation of why the sky is blue, to Rutherford's discovery of the atomic nucleus, through the modern medical applications of computerized tomography, scattering phenomena have attracted, perplexed and challenged some of the outstanding scientists and mathematicians of the twentieth century. Broadly speaking, scattering theory is concerned with the effect an inhomogeneous medium has on an incident particle or wave. In particular, if the total field is viewed as the sum of an incident field u^i and a scattered field u^s then the direct scattering problem is to determine u^s from a knowledge of u^i and the differential equation governing the wave motion. Of equal (or even more) interest is the inverse scattering problem of determining the nature of the inhomogeneity from a knowledge of the asymptotic behavior of u^s, i.e., to reconstruct the differential equation and/or its domain of definition from the behavior of (many) of its solutions. The above oversimplified description obviously covers a huge range of physical concepts and mathematical ideas, and for a sample of the many different approaches that have been taken in this area the reader can consult the monographs of Bleistein [**1**], Colton and Kress [**3**], Jones [**5**], Lax and Phillips [**8**], Newton [**9**], Reed and Simon [**10**], and Wilcox [**12**].

The simplest problems in scattering theory to treat mathematically are those of time harmonic acoustic waves which are scattered by either a penetrable inhomogeneous medium of compact support or by a bounded impenetrable obstacle. In addition to their appearance in realistic physical situations (e.g., acoustic tomography and nondestructive testing) such problems also serve as models for more complicated wave propagation problems involving electromagnetic waves, elastic waves, or particle scattering. To mathematically model these two problems, assume the incident field is given by the time harmonic plane wave

$$u^i(\mathbf{x}, t) = \exp[ik\mathbf{x} \cdot \alpha - i\omega t]$$

where $k = \omega/c$ is the wave number, ω the frequency, c the speed of sound and α the direction of propagation. Then the direct scattering problem for the case of an inhomogeneous medium is to find the total field $u = u^i + u^s$ such that

$$\Delta_3 u + k^2 n(\mathbf{x}) u = 0 \quad \text{in } R^3 \tag{1a}$$

$$u(\mathbf{x}) = \exp[ik\mathbf{x} \cdot \alpha] + u^s(\mathbf{x}) \tag{1b}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{1c}$$

where $n(\mathbf{x})$ is the (known) ratio of the square of the sound speeds in the homogeneous and inhomogeneous medium and (1c) is the Sommerfeld radiation condition which guarantees that the scattered wave is "outgoing". On the other hand, if it is assumed that the scattering is due to a given impenetrable "sound-soft" obstacle D then the task is to find the total field $u = u^i + u^s$ such that

$$\Delta_3 u + k^2 u = 0 \quad \text{in } R^3 \setminus \overline{D} \tag{2a}$$

$$u(\mathbf{x}) = \exp[ik\mathbf{x} \cdot \alpha] + u^s(\mathbf{x}) \tag{2b}$$

$$u = 0 \quad \text{on } \partial D \tag{2c}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0. \tag{2d}$$

We note that boundary conditions other than (2c) are possible (and indeed often more physically realistic) and in the following when we refer to problem (2), we shall always include the possibility of boundary conditions different from (2c). Although problems (1) and (2) are perhaps the simplest examples of physically realistic problems in scattering theory, they still cannot be considered completely solved, particularly from a numerical point of view, and remain the subject matter of much ongoing research.

The above two problems are both direct scattering problems. As already mentioned, the inverse scattering problem is of at least equal interest and is often far more challenging mathematically. In order to formulate the inverse scattering problems associated with problems (1) and (2), note that if u^s is the scattered field then u^s has the asymptotic behavior

$$u^s(\mathbf{x}) = \frac{e^{ikr}}{r} F(\hat{\mathbf{x}};\, k, \alpha) + O\left(\frac{1}{r^2}\right) \tag{3}$$

where $r = |\mathbf{x}|$, $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$ and $F(\hat{\mathbf{x}};\, k, \alpha)$ is known as the *far field pattern* or *scattering amplitude*. Then the inverse scattering problem is to determine either $n(\mathbf{x})$ or D from a knowledge of $F(\hat{\mathbf{x}};\, k, \alpha)$ for $\hat{\mathbf{x}}$ and α on some subset of the unit sphere and (possibly) different values of k. The area of inverse scattering theory is presently experiencing a rapid period of growth and, together with the corresponding direct scattering problems, is the subject matter of the book under review.

The mathematical methods used to investigate the direct and inverse scattering problems depend heavily on the frequency of the wave motion. In particular, if the wavelength $\lambda = 2\pi/k$ is very small compared with the smallest distance which can be observed with the available apparatus, the scattering obstacle produces a shadow with an apparently sharp edge. Closer examination reveals that the edge of a shadow is not sharply defined but breaks up into fringes. This phenomenon is known as diffraction. At the other end of the scale, obstacles which are small compared with the wavelength disrupt the incident wave without producing an identifiable shadow. Hence, we can distinguish three different frequency regimes corresponding to the magnitude of ka where a is a typical dimension of the scattering object. More specifically, the set of values of k such that $ka \ll 1$ is called the Rayleigh region, k such that $ka \gg 1$ is called the high-frequency regime, and the set of intermediate values of k is called the resonance region. As suggested by the observed physical differences, the mathematical methods used to study scattering phenomena in the Rayleigh or resonance region differ sharply from those used in the high-frequency regime.

The first question to ask about the direct scattering problem is that of uniqueness. The basic tools used to establish uniqueness are Green's formula and the unique continuation property of solutions to elliptic equations. Since equation (2a) has constant coefficients, the uniqueness question for problem (2) is the easiest to handle and was first treated by Sommerfeld in 1912. Subsequent generalizations were then given by Rellich, Vekua and Wilcox, all under the assumption that $\operatorname{Im} k \geqslant 0$. Analogous results for the case of problem (1) came later when Müller established the unique continuation property for solutions to (1a). When $\operatorname{Im} k < 0$, there can exist values of k for which uniqueness no longer holds. Such values of k are called *resonance states* and have fascinated researchers in scattering theory for over a hundred years. In particular, the resonance states are intimately involved with the asymptotic behavior of the time dependent wave equation and in addition provide a characteristic "signature" of the scattering object which could hopefully be used to solve the inverse scattering problem. For excellent, but somewhat dated, surveys of the role played by resonance states in scattering theory, the reader is referred to Dolph [**4**] and Lax and Phillips [**8**]. Finally, we note that special problems occur when D is unbounded or $1 - n(\mathbf{x})$ no longer has compact support, and for a discussion of some aspects of this situation, we refer the reader to Reed and Simon [**10**] and Chapter VII of the book under review.

Having established uniqueness, the next question to turn to is existence and the numerical approximation of the solution. The most popular approach to existence has been through the method of integral equations. In particular, for problem (1), it is easily verified that for all positive values of k, u is the unique solution of the Fredholm integral equation

$$(4) \qquad u(\mathbf{x}) = \exp[ik\mathbf{x} \cdot \alpha] - k^2 \iint_B \Phi(\mathbf{x},\mathbf{y}) m(\mathbf{y}) u(\mathbf{y})\, d\mathbf{y}$$

where $m(\mathbf{x}) = 1 - n(\mathbf{x})$, B is the support of $m(\mathbf{x})$ and $\Phi(\mathbf{x},\mathbf{y})$ is the (normalized) fundamental solution to the Helmholtz equation (2a) satisfying the Sommerfeld radiation condition. However, the application of integral equation methods to problem (2) is more subtle. To see why this is so, suppose we look for a solution of problem (2) in the form of a double-layer potential

$$(5) \qquad u^s(\mathbf{x}) = \int_{\partial D} \psi(\mathbf{y}) \frac{\partial}{\partial \nu(\mathbf{y})} \Phi(\mathbf{x},\mathbf{y})\, ds(\mathbf{y}); \qquad \mathbf{x} \in R^3 \setminus \overline{D}$$

where ν is the unit outward normal to ∂D and ψ is a continuous density to be determined. Then, letting $\mathbf{x}$ tend to ∂D, it can be shown that ψ is a solution of a Fredholm integral equation of the second kind. Unfortunately, this integral equation is not uniquely solvable if k^2 is a Neumann eigenvalue of the Laplacian in D. This difficulty was first observed and resolved by Kupradze, Vekua, Weyl and Müller. A more direct approach to the problem was initiated by Werner in 1962 who suggested modifying the kernel in the representation (5) to include further source terms. This idea was further developed by Leis, Panich, Brakhage and Werner, Ursell and Jones and at present there are a variety of integral equation formulations of problem (2) that are uniquely solvable for all (positive) values of the wave number. For a survey of these methods, the reader is referred to the article of Kleinman and Roach [7], the monograph of Colton and Kress [3] and the present book by Ramm.

The method of integral equations, as applied to problem (2), is particularly attractive from a numerical point of view since it not only reduces the dimensionality of the problem but also allows one to replace a problem over an unbounded domain by one over a bounded domain. Having done this, it is now possible to use a variety of numerical methods for solving Fredholm integral equations of the second kind. However, if ka is too large, the integrands become highly oscillatory and the method is not practical for numerical computation. On the other hand, if ka is small, the integral equation can be solved by iterative methods and this approach has been extensively developed by Kleinman and his co-workers. Indeed, as first observed by Rayleigh in 1897, for small values of ka a first approximation to the solution can be obtained without solving any integral equation at all.

Another popular method for solving problem (2) is through the use of the separation-of-variables solution to (2a), (2d) given by

$$(6) \qquad v_{nm}(\mathbf{x}) = h_n^{(1)}(k|\mathbf{x}|) Y_n^m(\hat{\mathbf{x}}); \qquad n = 0, 1, 2, \ldots, \quad -n \leqslant m \leqslant n,$$

where $h_n^{(1)}$ denotes a spherical Hankel function and Y_n^m a spherical harmonic. Since the set $\{v_{nm}\}$ is complete in $L^2(\partial D)$, a simple approach to constructing an approximate solution to problem (2) is to use (6) to perform a least squares fit to the boundary data of u^s on ∂D. A more sophisticated approach is the null-field or T-matrix method proposed by Waterman in 1965. In this approach, one uses Green's formula and the addition formula for Bessel functions to arrive at a generalized moment problem for the determination of the unknown function $\partial u^s/\partial \nu$. Having solved this problem for $\partial u^s/\partial \nu$, an application of Green's formula then gives $u^s(\mathbf{x})$ for $\mathbf{x} \in R^3 \setminus \overline{D}$. Note that the problem of interior eigenvalues that appears in the method of integral equations is not a problem for methods based on the complete set (6). However, from a numerical point of view, these methods are again restricted to values of ka in the Rayleigh or resonance region.

If ka is in the high-frequency region, the classical approach for solving problems (1) and (2) is geometrical optics. However, this method fails to account adequately for the behavior of the field in shadow regions and near edges. To cope with these problems, an extended version of geometrical optics known as the geometrical theory of diffraction was introduced by Keller in 1953. This approach is based on solving a set of ray equations and transport equations supplemented by certain canonical problems near edges and curved boundaries and yields a formal asymptotic expansion of the desired solution. The geometrical theory of diffraction has become the standard method for solving high-frequency scattering problems, and for a lucid description of this theory, we refer the reader to Keller's survey article [**6**]. A rigorous asymptotic analysis of the solution to problems (1) and (2) remains a formidable task, although significant progress has been made by Ursell, Morawetz, Ludwig, Bloom, Kazarinoff, Taylor, Majda, and Alber.

As indicated by the above discussion, the direct scattering problem has been thoroughly investigated and various efficient methods are available for its solution. In contrast, the inverse scattering problem of determining $n(\mathbf{x})$ or D from a knowledge of the far field pattern is still poorly understood and until recently has consisted mainly of a collection of ad hoc techniques with little rigorous mathematical basis. The reason for this is that the inverse scattering problem is inherently nonlinear and improperly posed. In particular, small perturbations of the far field pattern in any reasonable norm lead to a function which lies outside the class of far field patterns. Nevertheless, due to the demands of scientists in areas such as radar, sonar, geophysical exploration, medical imaging and nondestructive testing, various methods for solving the inverse scattering problem have been mathematically investigated and are beginning to be numerically implemented. As with the direct scattering problem, the methods vary according to which frequency regime is used to probe the unknown medium. However, due to the problem of poor resolution in the Rayleigh region, the only frequency regimes of practical interest are the resonance and high-frequency regions. Surveys of the inverse scattering problem have recently been given by Sleeman [**11**] and Colton [**2**].

In the case of high frequencies, the main methods used to investigate the inverse scattering problem for (1) are based on the Born and Rytov approximations where it is assumed that $m(\mathbf{x}) = 1 - n(\mathbf{x})$ is small. In particular, the Born approximation is just the first term in the Neumann series solution of (4) and hence is a valid approximation for $k^2 m(\mathbf{x})$ small. These approaches have been used by Bleistein and Cohen to investigate the seismic inverse problem and by Devaney to develop his theory of acoustic tomography. On the other hand, for problem (2) the main methods used to solve the high-frequency inverse scattering problem are based on the Kirchhoff, or physical optics, approximation

$$(7) \qquad F(-\alpha;\ k, \alpha) = 2ik \int_{\partial D^+} e^{2ik\mathbf{x}\cdot\alpha} \nu \cdot \alpha \, ds$$

where $F(-\alpha;\ k, \alpha)$ is the "backscattered" far field date and ∂D^+ is the part of ∂D illuminated by the incident field $e^{ik\mathbf{x}\cdot\alpha}$, i.e. $\partial D^+ = \{\mathbf{x} \in \partial D\colon\ \alpha \cdot \nu \geqslant 0\}$. Beginning with Bojarski in 1967, many mathematicians have used (7) to propose various algorithms for solving the inverse scattering problem. A rigorous justification of (7), together with a deep theoretical discussion of the inverse scattering problem, was undertaken by Majda in 1976 with subsequent contributions being given by Lax, Phillips, Majda, and Taylor in 1977.

In some situations (for example, when the above approximations are not applicable or in order to achieve higher penetrability of the probing field), it is desirable to study the inverse scattering problem for frequencies in the resonance region. In this case, recourse is usually made to constrained nonlinear optimization techniques. In particular, for the case of problem (2), it was recently shown by Colton, Kirsch and Monk that for fixed k the far field patterns corresponding to different directions of the incident field are all clustered around a hyperplane in $L^2(\partial\Omega)$, where $\partial\Omega$ is the unit sphere. The normal to this hyperplane is, of course, a function of D. Hence, one has two distinct optimization schemes for finding D from the far field data:

(1) Find D whose far field pattern best fits the measured data, and

(2) Find D whose associated normal vector in $L^2(\partial\Omega)$ is orthogonal to the measured far field data.

Both of these optimization methods are currently being investigated by many people in the scattering community, particularly from the viewpoint of numerical efficiency and flexibility. Similar projection methods have also recently been developed for the inverse scattering problem corresponding to problem (1).

The monograph under review is basically a description, with full details, of the use of integral equation methods in time harmonic acoustic wave propagation. As explicitly stated in the preface, the author has placed particular emphasis on his own contributions to the area, to the extent that most of the chapters are mainly a description of the author's own work. This has the unfortunate consequence that closely related work by other mathematicians is often mentioned only in passing, thus giving a somewhat distorted view of current research in the area of scattering theory. This is reflected in the bibliography, where roughly a third of all the references are to the author. Hence, the statements in the series editor's preface that this book "has practically zero intersection with all books I know about direct and inverse scattering" and "most of what [the book] contains can only be found in the research journal literature and is exposed here for the first time within the larger framework of a coherent book" should be interpreted in the light of the above comments. On the other hand, the author is a leading authority in the area of scattering theory and much of what he has to say is important. In particular, there is a wealth of valuable information in Ramm's book that cannot be found elsewhere. In addition, the book is clearly written and attractively presented. The previously stated reservations aside, Ramm's book is a valuable contribution to the scattering theory literature and will serve as an important reference for many years to come.

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Eigenvalues in Riemannian geometry, by Isaac Chavel, with a chapter by Burton Randol and an appendix by Jozef Dodziuk, Academic Press, Inc. (Harcourt Brace Jovanovich, Publishers), Orlando, San Diego, New York, London, Toronto, Montreal, Sydney, Tokyo, 1984, xiv + 362 pp., $62.00. ISBN 0-12-170640-0

The recent explosion of activity studying the relation between geometric and analytic properties of spaces has fused many areas of mathematics, such as the traditionally disparate fields differential geometry, partial differential equations, topology, mathematical physics, and number theory. One of the most popular topics in this study is the search for properties of the spectrum of the Laplace operator of a manifold in terms of its geometric invariants. Until recent decades there have been few significant developments, owing to the need for expertise in many fields. Eigenvalue problems are directly related to many geometric problems as well as to the disciplines mentioned above. Moreover, the techniques that have been developed in studying the Laplacian and its spectrum are equally important as the theorems about eigenvalues. This versatility factor coupled with the recent undeniable success of geometric analysis is responsible for the sudden blossoming of this classical area of mathematics.

The most fundamental object of study is the Laplace-Beltrami operator. Being invariantly defined, it is the simplest geometric elliptic operator which appears everywhere in geometry. It is the principal part of the expression for scalar curvature of a conformal factor in a metric as well as the mean curvature and stability form of a hypersurface. More importantly it is the linearization of the many nonlinear operators in geometry such as the Gauss curvature operator, the mean curvature operator and the Monge-Ampère operator. It is