who enjoyed studying *Complex manifolds* it is worth noting that *Complex manifolds and deformation structures* is a considerably expanded version of parts of that book.

Certainly those interested in complex geometry (whether algebraic, analytic, or differential geometric) will want a copy of *Complex manifolds and deformation of complex structures* on their bookshelf.

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Semi-Riemannian geometry: With applications to relativity, by Barrett O'Neill, Pure and Applied Mathematics, vol. 103, Academic Press, 1983, xiii + 457 pp., $45.00. ISBN 0-12-526740-1

This book gives a thorough introduction to semi-Riemannian (i.e., pseudo-Riemannian) manifolds using the notation of modern differential geometry. The author assumes that the reader has some knowledge of point-set topology, but does not assume a background in differential geometry. Thus, the book begins with a good introduction to smooth manifolds and tensor fields on manifolds.

If M is a smooth manifold, then a semi-Riemannian metric tensor g is a symmetric nondegenerate $(0,2)$ tensor field on M of constant signature. In local coordinates $x_1, x_2, \ldots, x_n$ one writes

$$ds^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij}\, dx_i\, dx_j.$$

Locally g is represented by the symmetric matrix $(g_{ij} = g_{ij}(x_1, \ldots, x_n))$ of smooth functions of x. This tensor is required to have a constant number s of negative eigenvalues and $n - s$ positive eigenvalues. The index s is zero for Riemannian (i.e., positive definite) manifolds, and for Lorentzian manifolds

$s = 1$ and $n \geqslant 2$. In general, $1 \leqslant s \leqslant n$ and there is a duality between spaces of index s and spaces of index $n - s$ which comes from replacing a given metric g_0 on M with $-g_0$.

For Riemannian manifolds, the positive definiteness of the metric tensor implies there is a Euclidean structure induced on each tangent space. Thus, the unit sphere in each tangent space is compact. Furthermore, one may define from g a distance function d: $M \times M \to \mathbf{R}$ which is a metric in the usual sense of metric spaces. Moreover, for *any* Riemannian metric g, the given manifold topology and the metric topology induced on M by d coincide. When the metric tensor g has index $1 < s < n$, then there is no naturally induced function on all of $M \times M$ which is a true distance.

Lorentzian manifolds are the geometric structures used to model gravity in Einstein's theory of General Relativity. The metric tensor for a Lorentzian manifold defines a Minkowskian (i.e., special relativistic) metric on each tangent space. This corresponds to Einstein's assumption in General Relativity that near each point of space-time, the laws of Special Relativity should be approximately true. For these manifolds, the set corresponding to the unit sphere in the tangent space is not compact.

The author emphasizes Riemannian and Lorentzian manifolds, but also develops the basic tools for semi-Riemannian manifolds of arbitrary index. He gives a clear discussion of the Levi-Civita connection, geodesics, the exponential map, curvature, and the geometric theory of (nondegenerate) submanifolds. He also considers covering manifolds, submersions, and orientability. All of these topics are familiar to readers who have studied the differential geometry of positive definite spaces, and these readers will find these parts of the semi-Riemannian theory to be quite similar to the Riemannian theory.

For Lorentzian manifolds, there is a second type of orientability called time orientability. Here (M, g) is time orientable iff there is a continuous vector field X on M satisfying $g(X, X) < 0$ at all points. In physics, a time oriented Lorentzian manifold is called a space-time, and the vector field X is used to determine the future and past at each point. These manifolds have an important induced partial ordering which is called the causal structure. A third type of orientability considered is called space orientability. This is orientability for spacelike directions and is thus the complementary notion to time orientation.

Let $\mathbf{R}^n_s$ denote $\mathbf{R}^n$ with a diagonal metric tensor represented by $(\delta_{ij}\varepsilon_j)$ where $\varepsilon_1 = \cdots = \varepsilon_s = -1$ and $\varepsilon_{s+1} = \cdots = \varepsilon_n = +1$. This is the semi-Euclidean space of index s and the isometries of this space holding the origin fixed form the semiorthogonal group $O(s, n - s)$. O'Neill has an interesting chapter on isometries of semi-Riemannian manifolds describing semiorthogonal groups and their Lie algebras. Also discussed are Killing fields (i.e., infinitesimal isometries), homogeneous spaces where each point may be mapped to any other by an isometry, and space forms which are complete connected semi-Riemannian manifolds of constant curvature.

One strong point of this book is the discussion of warped products. Given two semi-Riemannian manifolds (M, g) and (H, h) as well as a function f: $M \to \mathbf{R}^+$, a warped product metric on $M \times H$ is defined by the tensor

$g \oplus (fh)$. Similarly, one may define warped product metrics of the form $(eg) \oplus h$ on $M \times H$ where e: $H \to \mathbf{R}^+$. These warped product metrics form an interesting generalization of Cartesian product metrics, or from another viewpoint, surfaces of revolution. The warped product construction was used by Bishop and O'Neill to construct complete Riemannian manifolds of negative curvature. Also, in General Relativity, warped products are particularly important because all of the well-known classical solutions of the Einstein field equations can be expressed as warped products.

The mathematician with an interest in physics will find this an informative book which contains a number of physical discussions designed to motivate the development of Lorentzian manifolds. There is a chapter on Special Relativity where one uses the flat Minkowski metric $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ on $\mathbf{R}^4$. In this chapter the author considers velocity addition, the famous twin paradox, and Lorentz contraction. Furthermore, geometric units are discussed. In these units, the speed of light and the Newtonian gravitational constant are both taken to be unity. A chapter on cosmology is included in this book. Here the "big bang" Robertson-Walker metrics are studied. These are warped products of the form $(-dt^2) \oplus f(t)d\sigma^2$ on $I \times S$, where I is an open interval of the real axis and $(S, d\sigma^2)$ is a space of constant curvature. This book also has a chapter on the Schwarzschild solution. This is the metric used to model nonrotating black holes as well as spherically symmetric bodies such as stars and planets. The Schwarzschild solution is an example of a standard static space-time, i.e., a warped product metric of the form $(-fdt^2) \oplus h$ on $I \times H$, where (H, h) is Riemannian and f: $H \to \mathbf{R}^+$.

An important feature of this book is the inclusion of two of the various singularity theorems of Hawking and Penrose. These theorems show that under certain conditions involving the Ricci curvature and the convergence of geodesics, the given space-time contains an incomplete causal geodesic. Here a geodesic is said to be incomplete if it cannot be extended to have its maximal domain of definition equal to all of $\mathbf{R}$. This incomplete geodesic corresponds to the existence of a singularity. Thus, the results of Hawking and Penrose are interpreted as evidence that Einstein's formulation of General Relativity cannot be continued across certain singularities which arise in black holes and at the "big bangs" of cosmology. The partial ordering (i.e., causality) of space-times mentioned above is crucial to the proof of the singularity theorems. Also, important to these results is the author's discussion of the first and second variations of arc length together with the index form. The author gives a good discussion of these objects and Jacobi fields as well as the associated concepts of conjugate points and focal points.

Another strong point of this book is the inclusion of a large number of important and useful problems. The book is written so that the readers with less time at their disposal may skip the exercise sections; however, most people will find these sections very helpful.

This is an excellent book for anyone interested in semi-Riemannian geometry and/or General Relativity. It fills an important niche in the literature.

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