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GENERALIZED EXPONENTS VIA HALL-LITTLEWOOD SYMMETRIC FUNCTIONS

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The generalized exponents of finite-dimensional irreducible representations of a compact Lie group are important invariants first constructed and studied by Kostant in the early 1960s. Their actual computation has remained quite enigmatic. What was known ([**K**] and [**Hs**, Theorem 1]) suggested to us that their computation lies at the heart of a rich combinatorially flavored theory.

This note announces several results all tied together by Theorem 2.3 below, which selects the natural generalizations of the Hall-Littlewood symmetric functions, rather than the irreducible characters, as the best basis of the character ring. Full details will appear elsewhere.

1. Statement of problem. Let $\mathfrak{g}$ be a complex semisimple Lie algebra with adjoint group G. Via the adjoint action, the symmetric algebra $S(\mathfrak{g})$ becomes a graded representation of G. Kostant studied this representation in his fundamental paper [**K**]; his results are well known. $S(\mathfrak{g}) = I \otimes H$ is a free module over the G-invariants I generated by the harmonics H. Moreover, I is a polynomial ring on homogeneous generators of known degrees, and $H = \bigoplus_{p\geq 0} H^p$ is a graded, locally finite $\mathfrak{g}$-representation.

Hence, to study the isotypic decomposition of $S(\mathfrak{g})$, one forms for each irreducible G-representation V the polynomial in an indeterminate q:

$$F(V) := \sum_{p\geq 0} \langle V, H^p \rangle q^p. \tag{1.1}$$

Here $\langle\ ,\ \rangle$ is the usual form dim $\mathrm{Hom}_{\mathfrak{g}}(\ ,\ )$ on the representation ring of $\mathfrak{g}$. Kostant's problem asks us to determine $F(V)$; he called the integers $e_1, \ldots, e_s$ with $F(V) = \sum_{i=1}^{s} q^{e_i}$ the *generalized exponents of* V.

The polynomial $F(V)$ turns out to be a rather deep invariant of the representation V. For instance, the $F(V)$ are certain Kazhdan-Lusztig polynomials for the affine Weyl group (combine [**Hs**, Theorem 1] and [**Ka**, Theorem 1.8]), and they describe a certain group cohomology [**FP**, Theorem 6.1]).

2. A bilinear form. Our idea is to interpret F as a bilinear form on the character ring Λ of $\mathfrak{g}$. Precisely, *define* a $\mathbf{Z}[q]$-valued symmetric bilinear form $\langle\langle\ ,\ \rangle\rangle$ on $\Lambda[q]$ by setting

$$\langle\langle \mathrm{ch}(V_1), \mathrm{ch}(V_2) \rangle\rangle := F(V_1 \otimes V_2^*), \tag{2.1}$$

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for any two $\mathfrak{g}$-representations V_1 and V_2, and extending q-bilinearly. (Here $\mathrm{ch}(V)$ and V^* mean the character and dual of V.) Our (2.1) makes sense as (1.1) actually defines F on any representation of $\mathfrak{g}$.

We will present a basis in which our new form $\langle\langle\ ,\ \rangle\rangle$ diagonalizes. First fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and some familiar associated objects. Let Φ be the root system with Φ^+ a choice of positive roots. Form the lattice $\mathcal{P}$ of integral weights and its subset $\mathcal{P}^{++}$ of dominant ones. Let W be the Weyl group with length function l. Set

$$t_\pi(q) := \sum_{\substack{w\in W\\ w\cdot\pi=\pi}} q^{l(w)}, \quad \text{for } \pi \in \mathcal{P}.$$

Use exponential notation for characters.

Define, for $\lambda \in \mathcal{P}^{++}$, the *Hall-Littlewood characters*

$$P_\lambda := t_\lambda(q)^{-1} \sum_{w\in W} w\left(e^\lambda \prod_{\varphi>0} \frac{1-qe^{-\varphi}}{1-e^{-\varphi}}\right). \tag{2.2}$$

These characters are the classical Hall-Littlewood symmetric functions (see [**M**, III]) when $\mathfrak{g} = \mathfrak{sl}_n$; they appear in this more general context in work of Kato [**Ka**].

THEOREM 2.3. *The P_λ, $\lambda \in \mathcal{P}^{++}$, form an orthogonal $\mathbf{Z}[q]$-basis of $\Lambda[q]$ with respect to the form $\langle\langle\ ,\ \rangle\rangle$, and*

$$\langle\langle P_\lambda, P_\lambda\rangle\rangle = t_0(q)/t_\lambda(q).$$

We prove this by comparing $\langle\langle\ ,\ \rangle\rangle$ to $\langle\ ,\ \rangle$ via the expansion $\sum_{p\geq 0} \mathrm{ch}(H^p)q^p = t_0(q)\prod_\varphi(1-qe^\varphi)^{-1}$, as we know [**G1**, Theorem 2.5] the basis of $\Lambda[[q]]$ dual to $\{P_\lambda\}_{\lambda\in\mathcal{P}^{++}}$ with respect to $\langle\ ,\ \rangle$.

Kato [**Ka**] expressed the irreducible characters $\chi_\pi = \mathrm{ch}(V_\pi)$, V_π the $\mathfrak{g}$-representation of highest weight $\pi \in \mathcal{P}^{++}$, in terms of the P_λ: $\chi_\pi = \sum_{\lambda\in\mathcal{P}^{++}} m_\pi^\lambda(q)P_\lambda$. The polynomials $m_\pi^\lambda(q)$ are Lusztig's q-analogs of λ-weight multiplicity in V_π [**L**]; they satisfy $m_\pi^\lambda(1) = \dim(V_\pi^\lambda)$. We get

COROLLARY 2.4. *For $\alpha, \beta \in \mathcal{P}^{++}$,*

$$F(V_\alpha \otimes V_\beta^*) = \langle\langle \chi_\alpha, \chi_\beta\rangle\rangle = \sum_{\theta\in\mathcal{P}^{++}} m_\alpha^\theta(q) m_\beta^\theta(q) t_0(q)/t_\theta(q).$$

As Kostant [**K**] proved $F(V)|_{q=1} = \dim(V^0)$ for all V, our formula is a "q-analog" of the fact

$$F(V_\alpha \otimes V_\beta^*)|_{q=1} = \sum_{\theta\in\mathcal{P}^{++}} \dim(V_\alpha^\theta)\dim(V_\beta^\theta)\#(W\cdot\theta).$$

3. Combinatorics of mixed-tensor SL_n-representations. We set $\mathfrak{g} = \mathfrak{sl}_n$ to illustrate the effective use of §2 in evaluating F on irreducibles.

We have formulated a stability theory (1981) for the generalized exponents based on a "mixed-tensor" parameterization $V_{\alpha,\beta}^{[n]}$ of the irreducible PGL_n-representations, using certain pairs α, β of partitions. First we discuss combinatorics of SL_n-representations.

The Weyl group of SL_n is the symmetric group S_n; Λ is the ring of symmetric functions in $x_i = \exp(t_i)$, $1 \leq i \leq n$, for t_i the coordinates on diagonal matrices in $\mathfrak{sl}_n$. $\mathcal{P}^{++}$ identifies with the set of partitions of less than n rows via $\sum_{i=1}^{n-1} c_i t_i \leftrightarrow (c_1, \ldots, c_{n-1})$. (Note $t_1 + \cdots + t_n = 0$.) Then, $\chi_\lambda = s_\lambda(x_1, \ldots, x_n)$, the classical *Schur function*, and $P_\lambda = P_\lambda(x_1, \ldots, x_n; q)$. See [**M**, I(3), III(2.6)] for the combinatorial theory.

Write partitions γ as nonincreasing sequences $\gamma = (\gamma_1, \gamma_2, \ldots)$, ignoring trailing zeroes, with *length* $l(\gamma) = \#\{i \mid \gamma_i \neq 0\}$ and *magnitude* $|\gamma| = \gamma_1 + \gamma_2 + \cdots = \text{degree}(s_\gamma)$. Also write $V_\gamma^{[n]}$, rather than V_γ.

Then $m_\lambda^\mu(q) = 0$ unless $|\lambda| - |\mu| = kn$, some k, in which case $m_\lambda^\mu(q) = K_{\lambda,\pi}(q)$, the *Kostka-Foulkes polynomial* attached to Young tableaux of shape λ and weight $\pi = \mu + (k^n)$, by [**M**, III, 6, Example 3].

Given *partitions* α *and* β with $l(\alpha) + l(\beta) \leq n$, we *defined* $V_{\alpha,\beta}^{[n]}$ as the Cartan piece in $V_\alpha^{[n]} \otimes (V_\beta^{[n]})^*$, i.e., the irreducible $\mathfrak{sl}_n$-component generated by the tensor product of the highest weight vectors in each factor. It follows that $V_{\alpha,\beta}^{[n]} = V_\gamma^{[n]}$ for γ the componentwise sum (put $s = l(\alpha)$, $t = l(\beta)$):

$$\gamma = \mathrm{prt}_n(\alpha, \beta) := \left(\alpha_1, \ldots, \alpha_s, \underbrace{0, \ldots, 0}_{n-s-t}, -\beta_t, \ldots, \beta_1 \right) + \left(\underbrace{\beta_1, \ldots, \beta_1}_{n} \right).$$

For example, $\mathbf{C} = V_{(0),(0)}^{[n]}$, and $\mathfrak{g} = V_{(1),(1)}^{[n]}$.

LEMMA 3.1. *Fix* $n \geq 1$. *Then the* $V_{\alpha,\beta}^{[n]}$, *where* α *and* β *satisfy* $l(\alpha) + l(\beta) \leq n$ *and* $|\alpha| = |\beta|$, *form an exhaustive, repetition-free list of the irreducible finite-dimensional representations of* PGL_n.

4. Stability for PGL_n harmonics. Stability was our original reason for foming the $V_{\alpha,\beta}^{[n]}$. Write H_n^p for the degree p harmonics.

THEOREM 4.1. *Fix* $p \geq 0$. *Then the number of irreducible* PGL_n*-components of* H_n^p *is constant for* $n \geq 2p$. *Moreover, the decomposition stabilizes:* $V_{\alpha,\beta}^{[n]}$ *occurs in* H_n^p *only when* $r = |\alpha| = |\beta| \leq p$, *and* $\langle V_{\alpha,\beta}^{[n]}, H_n^p \rangle$ *stabilizes for* $n \geq p + r$. *Thus, for some finite set* J^p *of partition pairs of common magnitude and some integers* $c_{\alpha,\beta}^p$,

$$H_n^p \simeq \bigoplus_{(\alpha,\beta)\in J^p} c_{\alpha,\beta}^p V_{\alpha,\beta}^{[n]}, \quad \text{for } n \geq 2p.$$

Our original proof worked by a combinatorial analysis of the pieces in $S(\operatorname{End} \mathbf{C}^n)$ using the Cauchy and Littlewood-Richardson rules. We, R. Stanley, and P. Hanlon then studied the stable series $\lim_{n\to\infty} F(V_{\alpha,\beta}^{[n]})$. See [**S**, **Hn**, and **G2**].

The key question raised by 4.1 is *the determination of the* $F(V_{\alpha,\beta}^{[n]})$ *as functions of two variables* q *and* n (with $n \geq l(\alpha) + l(\beta)$ always implicit).

For each value of n, $F(V_{\alpha,\beta}^{[n]}) \in \mathbf{Z}[q]$ is controlled by the partitions $\lambda = \mathrm{prt}_n(\alpha, \beta)$ and $\mu = (\beta_1^n)$ of magnitude $n\beta_1$. Precisely, $F(V_{\alpha,\beta}^{[n]}) = K_{\lambda,\mu}(q)$ (this follows by combining [**Hs**, Theorem 1] with [**M**, III, 6, Example 3]).

However, *in* §5 *we prove that* $F(V^{[n]}_{\alpha,\beta})$ *as a function of* q *and* n *is really "controlled" just by* α *and* β (symmetrically, as $F(V^{[n]}_{\alpha,\beta}) = F(V^{[n]}_{\beta,\alpha})$). Given a partition α, let $h_1(\alpha), \dots, h_r(\alpha)$ be its hook numbers and $\tilde{\alpha}$ its conjugate partition (see [**M**, I, 1]). Set $e(\alpha) := \sum_{i\geq 1} i\alpha_i$. Previously, we knew only

PROPOSITION 4.2. *Assume* $|\alpha| = r$.
(i) *If* $\beta = (1^r)$, *then*

$$F(V^{[n]}_{\alpha,\beta}) = q^{e(\tilde{\alpha})} \prod_{i=1}^{r} (1 - q^{n-r-\tilde{\alpha}_i+i})/(1 - q^{h_i(\alpha)}).$$

(ii) *If* $\beta = (r)$, *then* $F(V^{[n]}_{\alpha,\beta}) = s_\alpha(q, \dots, q^{n-1})$.

5. A formula for $F(V^{[n]}_{\alpha,\beta})$**.** Let us extend the $K_{\lambda,\mu}(q)$ to *skew-partitions* α/π (cf. [**M**, I, 1.5]). Although the latter are not partitions, they behave as such. The *skew-Schur function* is defined by $s_{\alpha/\pi} = \sum_\gamma \langle s_\pi s_\gamma, s_\alpha \rangle s_\gamma$. Now define $K_{\alpha/\pi,\theta}(q)$ as the coefficient of P_θ in $s_{\alpha/\pi}$. Set

$$b_\theta(q) := \prod_{i\geq 1} (1-q)\cdots(1-q^{m_i}), \quad \text{for } \theta = (i^{m_i}); \qquad b_{(0)} := 1.$$

THEOREM 5.1. *Fix* α *and* β *with* $|\alpha| = |\beta| = r$. *Then*

$$F(V^{[n]}_{\alpha,\beta}) = \sum_{\substack{\pi,\theta \\ |\pi|+|\theta|=r.}} (-1)^{|\pi|} K_{\alpha/\pi,\theta}(q) K_{\beta/\tilde{\pi},\theta}(q) \frac{(1-q^n)\cdots(1-q^{n-l(\theta)+1})}{b_\theta(q)}.$$

To prove this, we express $V^{[n]}_{\alpha,\beta}$ in terms of the $V^{[n]}_\gamma \otimes (V^{[n]}_\delta)^*$ using essentially a formula of Littlewood, and then apply 2.4.

Theorem 5.1 leads to new, unified proofs of several old results, among them 4.1, 4.2, and the stable theorem [**S**, 8.1] proved by Stanley. But mainly, 5.1 gives the first real means for computing the $F(V^{[n]}_{\alpha,\beta})$.

COROLLARY 5.2. *For some polynomial* $g^{\alpha,\beta}(q,z)$ *over* $\mathbf{Z}$,

$$F(V^{[n]}_{\alpha,\beta}) = \frac{g^{\alpha,\beta}(q, q^{n-r+1})}{(1-q)\cdots(1-q^r)}.$$

Moreover,

$$\frac{g^{\alpha,\beta}(q,z)}{(1-q)\cdots(1-q^r)} = \sum_{i=0}^{r} c_i(q) \frac{(1-q^{r-1}z)\cdots(1-q^{r-i}z)}{(1-q)\cdots(1-q^i)},$$

for some $c_i(q) \in \mathbf{Z}[q]$.

We have some conjectures on the form of the $g^{\alpha,\beta}(q,z)$. The examples below, done by hand, are new; the first is an old conjecture. Define

$$\begin{bmatrix} c_1 & \cdots & c_r \\ d_1 & \cdots & d_r \end{bmatrix}_q := \frac{(1-q^{c_1})\cdots(1-q^{c_r})}{(1-q^{d_1})\cdots(1-q^{d_r})}, \quad \text{for } c_i, d_i \in \mathbf{Z}^+.$$

We refrain from thinking about these unless they are polynomials in q.

EXAMPLE 5.3. If $\alpha = \beta = (2,1)$, then 5.1 yields

$$F(V_{\alpha,\beta}^{[n]}) = q^3 \begin{bmatrix} n+1 & n-1 & n-3 \\ 1 & 1 & 3 \end{bmatrix}_q + q^5 \begin{bmatrix} n-1 & n-2 & n-3 \\ 1 & 1 & 3 \end{bmatrix}_q .$$

EXAMPLE 5.4. Let us find $F(V_\gamma^{[6]})$ when $\gamma = (6,4,1,1)$. Then $\gamma = \mathrm{prt}_6(\alpha,\beta)$, for $\alpha = (4,2)$ and $\beta = (2,2,1,1)$. 5.1 gives

$$\begin{aligned} F(V_{\alpha,\beta}^{[n]}) = q^9 & \begin{bmatrix} n+2 & n+1 & n-1 & n-2 & n-4 & n-5 \\ 1 & 1 & 2 & 2 & 4 & 5 \end{bmatrix}_q \\ & + q^{12} \begin{bmatrix} n+2 & n-1 & n-2 & n-3 & n-4 & n-5 \\ 1 & 1 & 2 & 2 & 4 & 5 \end{bmatrix}_q \\ & + q^{15} \begin{bmatrix} n-1 & n-2 & n-2 & n-3 & n-4 & n-5 \\ 1 & 1 & 2 & 2 & 4 & 5 \end{bmatrix}_q \\ & + (q^9 + q^{10} + q^{11}) \begin{bmatrix} n & n-1 & n-2 & n-3 & n-4 & n-5 \\ 1 & 1 & 2 & 2 & 2 & 5 \end{bmatrix}_q . \end{aligned}$$

So, at $n = 6$, $F(V_\pi^{[6]}) = 2q^9 + 3q^{10} + 7q^{11} + 9q^{12} + 13q^{13} + 13q^{14} + 15q^{15} + 12q^{16} + 11q^{17} + 7q^{18} + 5q^{19} + 2q^{20} + q^{21}$.

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REFERENCES

[**FP**] E. M. Friedlander and B. J. Parshall, *On the cohomology of algebraic and related finite groups*, Invent. Math. **74** (1983), 85–117.

[**G1**] R. K. Gupta, *Characters and the q-analog of weight multiplicity*, I.H.E.S. preprint M/86/6 (submitted for publication).

[**G2**] R. K. Gupta, *Stability of polynomial powers of* $\mathfrak{gl}_n$ (in preparation).

[**Hn**] P. Hanlon, *On the decomposition of the tensor algebra of the classical Lie algebras*, Adv. in Math. **56** (1985), 238–282.

[**Hs**] W. H. Hesselink, *Characters of the nullcone*, Math. Ann. **252** (1980), 179–182.

[**Ka**] S. Kato, *Spherical functions and a q-analogue of Kostant's weight multiplicity formula*, Invent. Math. **66** (1982), 461–468.

[**K**] B. Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. **85** (1963), 327–404.

[**L**] G. Lusztig, *Singularities, character formulas, and a q-analog of weight multiplicities*, Analyse et Topologie sur les Espaces Singuliers (II–III), Astérisque **101-102** (1983), 208–227.

[**M**] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Clarendon Press, Oxford, 1979.

[**S**] R. Stanley, *The stable behavior of some characters of* SL(n, C), Linear and Multilinear Algebra **16** (1984), 3–27.

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