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Hardy classes and operator theory, by Marvin Rosenblum and James Rovnyak, Oxford Mathematical Monographs, Oxford Univ. Press, New York and Clarendon Press, Oxford, 1985, xii + 161 pp., $39.95. ISBN 0-19-503591-7.

Hardy space theory has its classical origins in the work of G. H. Hardy and the brothers Riesz, but the modern origins of the subject begin with the theorem of A. Beurling in 1949. The Hardy space H^2 is defined to be the space of functions f analytic on the unit disk such that

$$\|f\|_2^2 = \sup\left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2\,dt \colon 0 \leqslant r < 1\right\} < \infty.$$

The theorem of Beurling asserts that any such f has an inner-outer factorization $f = bg$ where b is an inner function and g is an outer function. By definition an inner function is a function analytic on the unit disk whose nontangential boundary values have modulus 1 almost everywhere on the unit circle. An outer function can be defined as the solution of the extremal problem of finding the function g in H^2 that maximizes $|g(0)|$ among all functions with $|g(e^{it})|$ equal to a prescribed function on the boundary. Both inner functions and outer functions have finer structure; an inner function can be factored further as the product of a Blaschke product and a singular inner function while an outer function is characterized by having an integral representation of a certain form. It was recognized already by Beurling that this purely function-theoretic result has connections with operator theory. Indeed, from this theorem one can classify all the closed invariant subspaces for the operator M_z of multiplication by z on H^2. One also sees that the smallest closed invariant subspace containing a given function f is bH^2 where b is the inner factor of f, and thus f is cyclic if and only if f is outer.

It is now known how to put all this material in abstract purely operator-theoretic terms. The operator M_z is a concrete instance of a unilateral shift operator on an abstract Hilbert space H (i.e., a linear isometry S such that $\bigcap_{n \geqslant 0} S^n H = (0)$) having a cyclic vector. The cyclicity requirement can be dropped if one considers vector-valued versions of H^2. The theory is enriched by the fact that there are other interesting and seemingly different concrete instances of abstract shift operators besides M_z on H^2. The wandering-subspace proof of the Beurling theorem (specifically, the aspect of characterizing invariant subspaces), due to P. Halmos and independently to J. Rovnyak in the early 1960s, uses only Hilbert space geometry and makes perfectly good sense in the setting of abstract unilateral shifts of arbitrary multiplicity. As a consequence it picked up the matrix-valued version of Beurling's theorem done in the meantime by P. Lax.

The two main applications of this "abstract shift analysis" framework discussed in the book under review are interpolation and factorization. We reverse the order in the book and discuss factorization first. Two types of factorization are discussed, the inner-outer factorization of a function in H^∞ already mentioned and the spectral factorization of a nonnegative function on the unit circle. For simplicity we discuss the approach for the classical case, but the point is that this abstract approach works equally well for matrix- and operator-valued functions. In abstract form, one thinks of an H^∞ function f as an analytic S-Toeplitz operator $A = M_f$ of multiplication by f; here by "A is analytic S-Toeplitz" we simply mean that $AS = SA$. One can then apply the invariant subspace representation form of the abstract Beurling theorem to the subspace AH^2 to prove an abstract inner-outer factorization theorem for analytic S-Toeplitz operators. The problem of spectral factorization in general is the following: we are given a positive semidefinite operator function $W(e^{it})$ and seek a representation as $W(e^{it}) = A(e^{it})^* A(e^{it})$ where A is an operator-valued outer function. For the scalar case, by the classical theorem of Szegő such a factorization is possible if and only if $\log|W|$ is integrable on the unit circle; for the operator-valued case of course the solution cannot be so simple. To use the abstract framework for this problem, assume that W is bounded and introduce the Toeplitz operator $T = T_W : f \to P(Wf)$ on H^2; here P is the orthogonal projection of L^2 of the unit circle onto H^2, where H^2 is identified as a subspace of L^2 via nontangential limit boundary value functions. In abstract form, Rosenblum showed that Toeplitz operators T are characterized by the identity $(*)$ $T = S^*TS$. If $T = T_W$ and W is a positive semidefinite function, one easily sees that T is positive semidefinite as an operator on H^2. Then H^2 (or an appropriate completion if T does not have closed range) becomes a new Hilbert space H_T in the inner product $\langle x, y \rangle_T = \langle Tx, y \rangle$. Moreover, equation $(*)$ exactly says that S is an isometry S_T as an operator on H_T. One can then apply the abstract Beurling theorem to the pair (S_T, H_T) to get a more conventional model for the shift S_T. From the map which transforms H_T to the model space, one gets the outer factor A for W. The only delicate point is: when is the isometry S_T actually a shift, i.e., when do we have

$\bigcap_{n\geqslant 0} S_T^n H_T = (0)$? This is the criterion in the abstract setting corresponding to the classical log-integrability condition. While this abstract criterion is not computable in general, one can derive from it useful sufficient conditions (such as the Lowdenslager criterion and various comparison tests). While other authors (such as B. Sz.-Nagy and C. Foiaş [**12**]) have taken a similar approach to factorization, I think the results on factorization of this type reached their most refined form in the work of Rosenblum and Rovnyak from the early 1970s (an account of which is given in the book). I mention in particular the general classes of matrix or operator trigonometric polynomials, rational functions with prescribed poles, entire functions of some particular exponential type, and various classes of pseudomeromorphic functions. As corollaries of a general factorization theorem for a class $M(u, v)$, the authors get factorization theorems (many of them new at the time for the general operator-valued case) for each of these special classes in a unified way. In work of J. W. Helton and the reviewer, it is shown how other types of factorization (Wiener-Hopf and signed spectral factorization) can also be handled by the same abstract shift analysis approach; these are only mentioned together with references in the book.

While the results here are quite general, they are not as explicit in special cases. For example, one does not get the detailed multiplicative structure of matrix inner functions due to Potapov in the middle 1950s unless one introduces some ideas from operator model theory (see [**4**]). Also I. Gohberg and his collaborators [**2, 7**] have obtained more explicit factorization theorems for matrix polynomials and rational matrix functions by using a more detailed linear algebraic spectral approach.

The second area of application discussed in the book is interpolation. The Nevanlinna-Pick interpolation problem is to find a function analytic on the unit disk and bounded by 1 that takes on prescribed values at prescribed points in the disk. In the early part of this century, Pick obtained a matrix positive-definiteness test for solutions to exist and Nevanlinna obtained an iterative scheme for obtaining all solutions. Nevanlinna's approach essentially used the classical Schwarz lemma at each step. The connection with Hardy spaces and abstract shift analysis begins with a 1967 paper of D. Sarason; he obtained the solution as an application of what later would be known as the Sarason-Sz.-Nagy-Foiaş lifting theorem. This theorem connects the commutant of a Hilbert space contraction operator with the commutant of its isometric dilation. The contraction operator one takes to be $P_{\mathscr{S}} M_z|\mathscr{S}$ where $\mathscr{S}$ is the orthogonal complement of an appropriate invariant subspace for M_z in H^2; the isometric dilation then is simply M_z on H^2. Again by thinking of functions in terms of the multiplication operators they induce, one can use operator theory to do function theory. Rosenblum and Rovnyak take a more abstract approach and derive an operator interpolation result of Nudelman using the lifting theorem. From this general result they then derive classical results (e.g. Carathéodory-Fejer as well as Nevanlinna-Pick and both together) as special cases. Also included are their results on boundary interpolation of Loewner type from the late 1970s. Many of these results are nontrivial refinements of the classical versions and are easier and better than those obtained recently by others using purely classical methods.

The book includes two background chapters on Hardy classes of operator and vector functions. A somewhat novel approach is the use of Hardy-Orlicz spaces and harmonic majorants to define functions of bounded type and the Nevanlinna class for operator-valued functions. This is a quick way to do it and enables the authors to handle factorization of unbounded functions with their abstract bounded operator theory.

Interest in vectorial Hardy space theory as a tool for understanding operators was high in the 1960s and early 1970s. One highlight was the treatise of Sz.-Nagy and Foiaş **[12]** on the functional model for a completely nonunitary contraction; related model theories were being developed by de Branges and Rovnyak **[3]** and by Livsic and his school **[4]** in the Soviet Union. Since then, function-theoretic operator theory has evolved in other directions; now there is more emphasis on Bergman and other exotic function spaces rather than Hardy spaces, where the function theory is completely different. However, in the early 1970s, connections of the old operator model theory with systems theory were pointed out by such people as J. W. Helton, P. Dewilde, J. Baras and P. Fuhrmann **[1, 5, 9, 10]**. This has now evolved to the point that "H^∞-control" is an identifiable branch of control theory. An older area of application, originating in classical work of Wiener and Kolmogoroff, is the theory of stationary stochastic processes where the spectral factorization problem comes up in prediction theory **[8]**. The book under review should be a valuable reference for workers and students in all these areas, as well as to classical complex analysts who are open to studying what operator theory and functional analysis can do for their subject. Other recent books on Hardy spaces **[6, 11]** have focused exclusively on the scalar theory.

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