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SYMPLECTIC GROUPOIDS AND POISSON MANIFOLDS

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0. Introduction. A symplectic groupoid is a manifold Γ with a partially defined multiplication (satisfying certain axioms) and a compatible symplectic structure. The identity elements in Γ turn out to form a Poisson manifold Γ_0, and the correspondence between symplectic groupoids and Poisson manifolds is a natural extension of the one between Lie groups and Lie algebras.

As with Lie groups, under certain (simple) connectivity assumptions, every homomorphism of symplectic groupoids is determined by its underlying Poisson mapping, and every Poisson mapping may be integrated to a canonical relation between symplectic groupoids. On the other hand, not every Poisson manifold arises from a symplectic groupoid, at least if we restrict our attention to ordinary manifolds (even non-Hausdorff ones), so "Lie's third fundamental theorem" does not apply in this context.

Using the notion of symplectic groupoid, we can answer many of the questions raised by Karasev and Maslov [**9**, **10**] about "universal enveloping algebras" for quasiclassical approximations to nonlinear commutation relations. (I wish to acknowledge here that [**9**] already contains implicitly some of the ideas concerning Poisson structures and their symplectic realizations which were presented in [**18**].) In fact, the reading of Karasev and Maslov's papers was one of the main stimuli for the work described here. Following their reasoning, it seems that a suitably developed "quantization theory" for symplectic groupoids should provide a tool for studying nonlinear commutation relations which is analogous to the use of topology and analysis on global Lie groups in the study of *linear* commutation relations. Such a theory would also clarify the relation, mostly an analogy at present, between symplectic groupoids, star products [**2**], and the operator algebras of noncommutative differential geometry [**3**].

More immediately, the notion of symplectic groupoid unifies many constructions in symplectic and Poisson geometry; in particular, it provides a framework for studying the collection of all symplectic realizations of a given Poisson manifold.

A detailed exposition of these results will appear in [**4**]. Many of the details were worked out during a visit to the Université Claude-Bernard Lyon I. I would like to thank Pierre Dazord for his hospitality in Lyon, as well as for many stimulating discussions. The idea of introducing groupoids into symplectic geometry arose in the course of conversations with Marc Rieffel about operator algebras and the subsequent reading of J. Renault's thesis [**14**].

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1. Definitions. We recall that a *groupoid* is a set Γ equipped with a subset Γ_0 of identity elements, projections α ("source") and β ("target") from Γ to Γ_0, a multiplication operation $(x,y) \mapsto xy$ defined whenever $\beta(x) = \alpha(y)$, and an inversion operation $\iota\colon \Gamma \to \Gamma$, satisfying algebraic axioms generalizing those of a group [**7, 11, 14**].

If Γ is a C^∞ manifold, and all the other objects appearing in the definition above are C^∞ manifolds, submanifolds, and mappings, with α and β submersions, then Γ is called a *differentiable groupoid.* If Γ is also equipped with a symplectic structure Ω for which the submanifold $\Gamma_3 = \{(z,x,y) |\ z = xy\}$ is lagrangian in $(\Gamma,\Omega) \times (\Gamma,-\Omega) \times (\Gamma,-\Omega)$, then we call Γ a *symplectic groupoid.* (The closely related but less useful concept of "*-algebra in the symplectic category" was introduced in [**17**].)

2. The Poisson manifold of a symplectic groupoid. It is easy to show that the inversion mapping on a symplectic groupoid Γ is antisymplectic and that Γ_0 is lagrangian. With some more effort, one shows that $\alpha^*(C^\infty(\Gamma_0))$ and $\beta^*(C^\infty(\Gamma_0))$ are the centralizers of one another in the Poisson bracket Lie algebra $C^\infty(\Gamma)$. It follows from the theory of Poisson dual pairs [**18**] that there is a uniquely determined Poisson structure on Γ_0 for which the mappings α and β are Poisson and anti-Poisson respectively.

3. Examples. Two classes of examples generalize those in [**17**].

A. *Cotangent bundles.* Let G be any differentiable groupoid, and let G_3 be the submanifold $\{(z,x,y)|\ z = xy\}$ in $G \times G \times G$. Then the conormal bundle $\nu^* G_3$ is lagrangian in $T^*(G \times G \times G) = T^*G \times T^*G \times T^*G$, where T^*G has the canonical symplectic structure Ω_G; multiplying the cotangent vectors in the last two factors by -1 gives a lagrangian submanifold of $(T^*G, \Omega_G) \times (T^*G, -\Omega_G) \times (T^*G, -\Omega_G)$ which is Γ_3 for a symplectic groupoid structure on $\Gamma = T^*G$. (Γ_3 is also the wavefront set for convolution in the groupoid algebra [**3, 14**] of G.) Γ_0 turns out to be the conormal bundle ν^*G_0, with a Poisson structure for which the functions on ν^*G_0 which are linear on fibres form a Lie subalgebra of $C^*(\nu^*G_0)$. (Thus, the sections of the normal bundle νG_0 form a Lie algebra; this is just the Lie algebroid of Pradines [**13**].)

For example, if G is a Lie group (a differentiable groupoid with just one identity element), the Poisson manifold Γ_0 is just the dual space $\mathbf{g}^*$ of the Lie algebra of G, with its well-known Lie-Poisson structure. At the other extreme, if G is a trivial groupoid (i.e. $G_0 = G$), then the groupoid structure on $\Gamma = T^*G$ is addition along the fibres, and the Poisson structure on $\Gamma_0 = G$ (the zero section in T^*G) is trivial. If G is the associated groupoid [**11**] of equivariant maps between fibres in a principal bundle $H \to B \to X$, then Γ_0 is the "phase space of a classical particle in a Yang-Mills field" [**15, 16**] with configuration space X and internal variables in $\mathbf{h}^*$. Finally, if G is the holonomy groupoid of a foliation [**3, 19**], then Γ_0 is the "cotangent bundle along the leaves", natural domain for the symbol calculus of the operator algebras associated with the foliation [**3**].

B. *Fundamental groupoids.* If (P,Ω) is any symplectic manifold, then $(P,\Omega) \times (P,-\Omega)$ is a symplectic groupoid with respect to the operation $(p,q)(q,r) = (p,r)$. Covering this product are the *fundamental groupoid* $\pi(P)$

[*homology groupoid* $\mathcal{H}(P)$] consisting of homotopy [homology] classes of paths in P with fixed endpoints. Both $\pi(P)$ and $\mathcal{H}(P)$ have compatible symplectic structures pulled up from $(P, \Omega) \times (P, -\Omega)$. Any symplectic action of a Lie group on P lifts to actions on $\pi(P)$ and $\mathcal{H}(P)$ which are *hamiltonian* (i.e., admitting ad*-equivariant momentum mappings).

4. Constructing symplectic groupoids. To construct a "local symplectic groupoid" Γ for a given Poisson manifold Γ_0, it is enough to have any symplectic manifold S equipped with a Poisson mapping α: $S \to \Gamma_0$ and a lagrangian cross section for α, which identifies Γ_0 with a submanifold of S. From this data, one can construct in a canonical way a local groupoid structure [**7**] on a neighborhood of Γ_0 in S.

It was shown in [**18**] that such a map α: $S \to \Gamma_0$ always exists if we take sufficiently small open subsets in Γ_0. Using the theory of symplectic groupoids, we can now show that these local symplectic realizations can be sewn together to produce a realization for all of Γ_0. This implies the existence of a local symplectic groupoid associated with every Poisson manifold. [*Added in proof*: This result, together with other ideas closely related to our work, is also contained in [**20**].]

It is not always possible to extend a local symplectic groupoid to a global one, though. Consider, for instance, $\Gamma_0 = S^2 \times \mathbf{R}$ with a Poisson structure in which the symplectic leaves are S^2's with area given by the function $t \mapsto 1+t^2$ on $\mathbf{R}$. Then the natural candidate for Γ turns out to be singular along a 4-dimensional manifold, with normal spaces given by the quotient of $\mathbf{R}^2$ by the $\mathbf{Z}$ action $n \cdot (t, y) = (t, y + nt)$. The nonexistence of a nonsingular Γ for this Γ_0 is related to the violation of the linear variation property of Duistermaat-Heckman [**6**]. The failure of "Lie III" and the need for generalized manifolds have already been observed in several related contexts [**1**, **5**, **7**, **8**, **12**].

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