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COUNTING LATIN RECTANGLES

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A $k \times n$ Latin rectangle is a $k \times n$ array of numbers such that each row is a permutation of $\{1, 2, \ldots, n\}$ and each column has distinct entries. The problem of counting Latin rectangles is of considerable interest. Explicit formulas for $k = 3$ are fairly well known [**1–3**, **4**, pp. 284–286 and 506–507, **5**, **6**, **9–11**, **12**, pp. 204–210]. Formulas for $k = 4$ were found by Pranesachar et al. [**1**, **9**] and a complicated formula for all k was found by Nechvatal [**8**]. We give here a simple derivation of a formula similar to Nechvatal's. The formula implies that for fixed k, the number of $k \times n$ Latin rectangles satisfies a linear recurrence with polynomial coefficients. We use properties of the Möbius functions of partition lattices, as did Bogart and Longyear [**2**], Pranesachar et al. [**1**, **9**], and Nechvatal [**8**], but in a somewhat different way.

In order to state the formula, we first make some definitions. Let $\mathcal{P}$ be the set of partitions of $\mathbf{k} = \{1, 2, \ldots, k\}$ and let S be the set of nonempty subsets of $\mathbf{k}$. If f is a function from $\mathcal{P}$ to the nonnegative integers $\mathbf{N}$, and A is in S, then we set $\langle f, A\rangle = \sum_{\pi \ni A} f(\pi)$, where the sum is over all partitions π of which A is a block. We shall say that two functions $f, g \colon \mathcal{P} \to \mathbf{N}$ are *compatible* if $\langle f, A\rangle = \langle g, A\rangle$ for each A in S.

THEOREM. *The number of $k \times n$ Latin rectangles is*

$$\sum_{f,g} \frac{n!^2}{\prod_{\pi \in \mathcal{P}} f(\pi)!\, g(\pi)!} \prod_{A \in S} (-1)^{\langle f,A\rangle(|A|-1)} (|A| - 1)!^{\langle f,A\rangle} \langle f, A\rangle!,$$

where the sum is over all compatible pairs f, g of functions from $\mathcal{P}$ to $\mathbf{N}$ *satisfying $\sum_{\pi \in \mathcal{P}} f(\pi) = \sum_{\pi \in \mathcal{P}} g(\pi) = n$.*

PROOF. We first restate the problem in terms of bipartite graphs. Given a $k \times n$ "rectangle" satisfying the row conditions, but with column entries not necessarily distinct, we may associate to it a bipartite graph with vertex sets $P = \{p_1, p_2, \ldots, p_n\}$ and $Q = \{q_1, q_2, \ldots, q_n\}$, and with edges colored in k colors. (We identify the set of colors with $\mathbf{k}$.) If the rectangle has the entry l

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in the (i, j) position, then the graph has an edge of color i from vertex p_j to vertex q_l. For simplicity we refer to bipartite graphs obtained from rectangles this way as *colored graphs.* They have the property that each vertex is incident with one edge of each of the k colors. A Latin rectangle will correspond to a colored graph without multiple edges. We call these graphs *Latin graphs.*

Let us define a *partitioned graph* to be an ordered pair (G, γ), where G is a colored graph and γ is a function defined on $P \times Q$ such that $\gamma(p, q)$ is a partition of the set of colors of the edges of G from p to g. We shall count Latin graphs by counting partitioned graphs weighted in such a way that the sum of the weights of all partitioned graphs corresponding to a given colored graph will be 1 if the colored graph is Latin and 0 otherwise.

By the Möbius function of a partition π of a set $T = \{t_1, t_2, \ldots, t_m\}$ we mean the Möbius function of the interval $[\hat{0}, \pi]$ in the lattice of partitions of T, where $\hat{0}$ is the partition $\{\{t_1\}, \{t_2\}, \ldots, \{t_m\}\}$. It is well known [**13**, **14**] that if π has blocks of sizes $b_1, b_2, \ldots, b_r$, then the Möbius function of π is $\prod_{i=1}^{r}(-1)^{b_i - 1}(b_i - 1)!$. Now let us weight a partitioned graph by the product of the Möbius functions of its partitions. It follows from the basic property of the Möbius function that, with this weighting, the sum of the weights of all partitioned graphs is equal to the number of Latin graphs.

We now need only count weighted partitioned graphs. If (G, γ) is a partitioned graph we can construct a new kind of object by "cutting in half" all the edges of G. When we do this we lose adjacencies of vertices, but we retain the partitions of colors at each vertex.

More formally, we define a *semigraph* to be a function α that assigns to each element of $P \cup Q$ a partition of the set of colors. The operation of "cutting the edges in half" applied to a partitioned graph (G, γ) assigns to it the semigraph α as follows: for $p \in P$,

$$\alpha(p) = \bigcup_{q \in Q} \gamma(p, q),$$

and for $q \in Q$,

$$\alpha(q) = \bigcup_{p \in P} \gamma(p, q).$$

If v is a vertex, we call the elements of $\alpha(v)$ the *half-edge blocks* at v.

We now construct a generating function for semigraphs. We shall then apply a linear operator that converts this generating function into the sum of the weights of the partitioned graphs.

For each $A \in \mathcal{S}$, let X_A and Y_A be variables, and for any partition π in $\mathcal{P}$, let

$$X_\pi = \prod_{A \in \pi} X_A \quad \text{and} \quad Y_\pi = \prod_{A \in \pi} Y_A.$$

We now assign to any semigraph α the weight

$$\prod_{p \in P,\ q \in Q} X_{\alpha(p)} Y_{\alpha(q)}.$$

It is clear that the sum of the weights of all semigraphs is

$$\left(\sum_{\pi\in\mathcal{P}} X_\pi\right)^n \left(\sum_{\sigma\in\mathcal{P}} Y_\sigma\right)^n. \tag{1}$$

Now given a semigraph α, how can we construct all partitioned graphs corresponding to it? We can do this by pairing up half-edge blocks of P with half-edge blocks of Q corresponding to the same set of colors. In order for this to be possible, it must be the case that for each subset A of $\mathbf{k}$, the number of half-edge blocks of P corresponding to A must equal the number of half-edge blocks of Q corresponding to A. If this is the case, and the number of each is c_A, then the half-edge blocks corresponding to A can be paired up in $c_A!$ ways, and each pair should be assigned the weight $(-1)^{(|A|-1)}(|A|-1)!$. Thus the number of Latin graphs is obtained by expanding (1), and then applying the linear operator that takes the monomial $\prod_{A\in S}(X_A Y_A)^{c_A}$ to

$$\prod_{A\in S} (-1)^{c_A(|A|-1)}(|A|-1)!^{c_A} c_A!$$

and takes monomials not of this form to zero. To get an explicit expression, we expand (1) by the multinomial theorem, getting

$$\sum_{f,g} \frac{n!^2}{\prod_{\pi\in\mathcal{P}} f(\pi)!g(\pi)!} \prod_{\pi\in\mathcal{P}} X_\pi^{f(\pi)} Y_\pi^{g(\pi)},$$

where the sum is over all functions $f, g\colon \mathcal{P} \to \mathbf{N}$ for which $\sum_{\pi\in\mathcal{P}} f(\pi) = \sum_{\pi\in\mathcal{P}} g(\pi) = n$, and the theorem follows easily.

It follows from the theorem and from results of Zeilberger [**16**] and Lipshitz [**7**] that for fixed k, the number $L_k(n)$ of $k\times n$ Latin rectangles is P-recursive [**15**], i.e., that for some M and for some polynomials $c_i(n)$, $i = 0,\dots,M$ (depending on k),

$$\sum_{i=0}^{M} c_i(n) L_k(n+i) = 0.$$

The method described here can also be used to count regular graphs and digraphs of various types, or equivalently, positive integer and 0-1 matrices with prescribed row and column sums.

REFERENCES

1. K. B. Athreya, C. R. Pranesachar, and N. M. Singhi, *On the number of Latin rectangles and chromatic polynomials of $L(K_{r,s})$*, European J. Combin. **1** (1980), 9–17.

2. K. P. Bogart and J. Q. Longyear, *Counting 3 by n Latin rectangles*, Proc. Amer. Math. Soc. **54** (1976), 463–467.

3. I. M. Gessel, *Counting three-line Latin rectangles*, Proc. Colloque de Combinatoire Énumérative, UQAM 1985, to be published.

4. I. P. Goulden and D. M. Jackson, *Combinatorial enumeration*, Wiley, 1983.

5. S. M. Jacob, *The enumeration of the Latin rectangle of depth three by means of a formula of reduction, with other theorems relating to non-clashing substitutions and Latin squares*, Proc. London Math. Soc. **31** (1930), 329–354.

6. S. M. Kerewala, *The enumeration of the Latin rectangle of depth three by means of difference equations*, Bull. Calcutta Math. Soc. **33** (1941), 119–127.

7. L. Lipshitz, *The diagonal of a D-finite power series is D-finite*, J. Algebra (to appear).

8. J. R. Nechvatal, *Asymptotic enumeration of generalized Latin rectangles*, Utilitas Math. **20** (1981), 273–292.

9. R. Pranesachar, *Enumeration of Latin rectangles via SDR's*, Combinatorics and Graph Theory, (S. B. Rao, ed.), Lecture Notes in Math., vol. 885, Springer-Verlag, Berlin and New York, 1981, pp. 380–390.

10. J. Riordan, *Three-line Latin rectangles*, Amer. Math. Monthly **51** (1944), 450–452.

11. ——, *Three-line Latin rectangles*. II, Amer. Math. Monthly **53** (1946), 18–20.

12. ——, *An introduction to combinatorial analysis*, Wiley, 1958.

13. G.-C. Rota, *On the foundations of combinatorial theory*, I. *Theory of Möbius functions*, Z. Wahrsch. Verw. Gebiete **2** (1964), 340–368.

14. M.-P. Schützenberger, *Contributions aux applications statistiques de la théorie de l'information*, Publ. Inst. Statist. Univ. Paris **3** (1954), 5–117.

15. R. P. Stanley, *Differentiably finite power series*, European J. Combin. **1** (1980), 175–188.

16. D. Zeilberger, *Sister Celine's technique and its generalizations*, J. Math. Anal. Appl. **85** (1982), 114–145.

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