it is more useful for learning the essentials of the subject than for gaining an overview of the state of the subject at the present time, despite the multitude of asides and context-placing references. The main thrust of the book is to work up from the idea of the index of a Fredholm operator on Hilbert space, through the introduction of pseudo-differential operators to the second proof of the index theorem, yielding the index in K-theory. The organization of the material is generally good, with theories and techniques brought in for the attainment of specific goals and not for their own sake. One slight hiccup in the linear organization is that K-theory gets defined twice, and if we include the gauge-theoretic section, Sobolev spaces on manifolds are defined twice also (and in different ways!). Nevertheless, the book provides a proof of the index theorem and a good description of what it can do.

Times move on, of course, and the more recent proofs of the index theorem which are motivated by supersymmetry provide a rationale for the role which functions like $x/(1 - e^{-x})$ and $x/\tanh x$ play in sorting out the combinations of characteristic classes which occur in the index formula. By now workers in partial differential equations, stochastic processes, Riemannian geometry, algebraic geometry, algebraic topology and mathematical physics all have the index theorem doing something for them. In another twenty years the list will almost certainly be longer.

Like Stonehenge, the theorem stands there as an immovable edifice, with each generation giving its own interpretation. For one it is a computational device, for another a more mystical representation of supersymmetry. Either way, it has created a bridge between mathematics and physics and has given mathematicians and physicists a deeper, or at least more sympathetic, understanding of each other's work. The Dirac operator will never be reinvented a third time!

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Sequence transformations and their applications, by Jet Wimp, Mathematics in Science and Engineering, vol. 154, Academic Press, 1981, xix + 257 pp., $38.50. ISBN 0-12-757940-0

In this volume the author restricts himself mostly to material on sequence transformations which has not appeared in book form in English. Some of the material is available in French (Brezinski, 1977, 1978), but much of the material has never appeared in book form in any language. Some has not appeared in published papers [the thesis work of Higgins (1976) and Germain-Bonne (1978) for instance], and much is new altogether.

The subject of this book touches virtually every area of analysis, including interpolation and approximation, Padé approximation, special functions, continued fractions, and optimization methods, to name a few.

Two types of sequence transformations are considered.

The first type is a transformation T mapping each sequence $s = \{s_n\} = (s_0, s_1, s_2, \dots)$ in a Banach space B (usually the complex field C or the real field R) into a sequence $t = T[s] = (t_0, t_1, t_2, \dots)$, also in B. The sequence s may be the nth partial sums of a series, i.e., $s_n = a_0 + a_1 + a_2 + \cdots + a_n$, $n \geqslant 0$, or may arise by various other means.

The main objective in the theoretical development of this book is to compare the convergence of t with that of s as concerns regularity and accelerativity of T. In particular, (1) if s converges to s^*, does t also converge to s^*, i.e., is T "regular" for s, and (2) if T is regular for s, does t "converge more rapidly" than s, i.e., does T "accelerate the convergence" of s, $d[t_n, s^*] = o(d[s_n, s^*])$. Here d is the metric induced by the norm in our Banach space B.

When considering acceleration from the theoretical viewpoint, the emphasis is not on individual sequences but rather on classes of sequences. Appropriately, if a transformation T accelerates the convergence of each sequence in such a class, T is said to accelerate that class.

In practice, a particular class of sequences may be initially at hand and a corresponding transformation T is devised by some scheme to accelerate this class. One problem then considered is to characterize the largest class, say "acc(T)," accelerated by T; but this may be difficult to do. This situation might well arise if T is "exact" for a certain class C^* of convergent sequences, which means that for each member s of C^* we have $t_n = s^*$, $n \geqslant 0$.

An associated problem which may be more tractable is: given two transformations T and T', determine if one of the two classes acc(T) and acc(T') is a subset of the other class. This problem becomes significant in computer applications when more than one transformation is available for accelerating the convergence of sequences and limited information about these sequences is available ahead of time.

In numerical applications, the interest is not in the fact that T accelerates the convergence of a sequence s (or a class of sequences), but rather in using one of the first few terms of the sequence $t = T(s)$ to obtain an acceptable approximation to s^*. This seems to be a somewhat paradoxical situation, since the condition that T accelerate the convergence of s implies nothing directly about the values of the first few terms of t. Of course, one way to deal with this situation for a real sequence $\{s_n\}$ is to obtain sharp upper and lower bounds on at least the first few terms of the sequence $\{s^* - s_n\}$. We will return to this situation near the end of the review.

In order to obtain an acceptable approximation to s^*, the numerical analyst typically may only use the values of t_n for, say, $n \leqslant 10$. This has at least been true without the availability of high-speed computers. Very often the t_n with the largest value of n, in our case $n = 10$, gives the best approximation of s^*.

If this approximation is not acceptable, the transformation T may be applied to t and the first few terms of $T[t]$ may be considered. This process may be repeated a few times until, hopefully, an acceptable approximation to s^* is obtained. Since only the first few terms of s are used, this repetitive process must cease after a small number of applications of the transformation T.

Rather startling results have been obtained by this repetitive process for "slowly convergent" sequences. In particular, the first few terms of a sequence s may not even approximate its limit s^* to one decimal place and the millionth term of s might be the first term to give an approximation accurate to, say, six decimal places. On the other hand, starting with only the first few terms of s and carrying out this repetitive process a few times, an approximation accurate to six decimals may well result. Consequently, the first few terms of s may inherently contain a considerable amount of information about s^* while each of these terms taken individually does not, and the problem is to extract this information from these first few terms.

We now turn to the second type of transformation which assigns to each sequence s a countable set of sequences $s(k,\ ) = (s(k,0), s(k,1), s(k,2), \dots)$, $k \geqslant 0$, by the use of a formula (called a "lozenge algorithm"). This set of sequences is used to form a double array $\{s(n,k)\}$, $n, k \geqslant 0$, in which the element $s(n,k)$ appears in the $(n+1)$th row and $(k+1)$th column of the array.

A lozenge algorithm of this type, called a "deltoid," is of considerable importance. For example, we may recursively set

$$s(n,k+1) = G[s(n+1,k), s(n,k)], \quad s(n,0) = s_n, \qquad n,k \geqslant 0.$$

This recursive relation simply states that the terms of the sequence s are used to fill out the first column of the array, and that an element in a given column of the array is determined by two elements from the preceding column and from the appropriate rows as prescribed by the relation. Here $G = G(x, y)$ is a function of two variables. In a similar situation with $G = G(x, y, z)$ as a function of three variables, a "rhomboid" is produced.

When dealing numerically with a transformation of the second type, a lozenge algorithm is sought which is "numerically stable," i.e., which will not break down in numerical applications because of the limited number of significant figures or decimal places that are being carried along in the computations.

Along this same line, it becomes important to consider the computing time and storage space required by each algorithm when one or more algorithms are available for a given transformation. Indeed, as the author points out, the Trench algorithm which is a variant of the BH protocol surprisingly requires only one-third as many operations as the ε-algorithm when used to generate the iterates $[e_k]_n = s(n,k)$ of the Schmidt transformation e_k, $k \geqslant 0$.

Regularity and acceleration are considered for various sequences t, each of which is made up of elements chosen from the array $\{s(n,k)\}$ along any so-called "path." This requires that $t_0 = s(0,0)$ and, having selected a term for t from a location in the array, the following term for t must be chosen from the next column to the right or the next row downwards relative to this location and, at the same time, cannot be chosen from any column to the left or any higher row relative to this location in the array. If, from some point on, the terms for t are all chosen from some fixed column (row) of the array, the path is called "vertical" ("diagonal"). The word "diagonal" is used since the rows of the array are displayed diagonally downwards.

Both linear and nonlinear transformations are treated.

The linear theory is initially based on the Toeplitz limit theorem specialized to transformations $T\colon CS \to CS\colon s \mapsto t$ which are linear transformations on CS, the space of complex sequences. In all practical situations for such transformations from CS to CS, the only possible linear transformation T may be defined by a lower triangular complex matrix $U = [u_{nk}]$, $u_{nk} = 0$, $k > n$, where $T[s] = t$ and

$$t_n = \sum_{k=0}^{n} u_{nk} s_k, \qquad n = 0, 1, 2, \ldots.$$

T or U is called a "triangle" if $\sum_{k=0}^{n} u_{nk} = 1$.

The polynomial

$$P_n(r) = \sum_{k=0}^{n} u_{nk} r^k, \qquad n \geqslant 0,$$

called the "nth characteristic polynomial" of the triangle U, and a useful function,

$$m(r) = \limsup |P_n(r)|^{1/n},$$

called the "measure of U," are shown to be intimately related to the regularity and accelerative properties of U.

The author discusses several important triangles, regular methods U applied to series with variable terms, and rational approximations. A number of linear lozenge methods are presented. Deltoids are investigated, along with quadrature applications. It is also shown how rhomboids can be developed for orthogonal triangles. Chebyshev and Legendre polynomials are given as examples. Richardson extrapolation and Romberg integration are also included.

Turning to the nonlinear theory, one cannot expect general lozenge algorithms to satisfy a theory as simple and elegant as the linear theory, which was made possible by the Toeplitz limit theorem.

Only recently has a beginning of a general theory been developed to encompass regularity and acceleration properties for various well-known nonlinear sequence transformations. Much of this work is due to Germain-Bonne (1973, 1978). It excludes some important algorithms and can only handle vertical convergence, i.e., convergence along vertical paths (Chapter 5).

As an example, this theory partially handles the accelerativity property of Aitken's δ^2-process A, defined by $A(s) = t$ where

$$t_n = \left(s_n s_{n+2} - (s_{n+1})^2\right) / (s_n - 2s_{n+1} + s_{n+2}) \tag{3}$$

or $t_n = s_n$ whenever the denominator of this fraction is zero.

Aitken's δ^2-process and related topics are given detailed attention (Chapter 7). Various results from Tucker (1967, 1969) are included, but I feel that it would have been appropriate to include the following theorem from Tucker (1969).

THEOREM. *Let s be a member of CS such that for some $r > 0$*

$$|a_{n+1}/a_n| \leqslant r < 1, \quad \textit{for } n \textit{ sufficiently large}, \tag{4}$$

where $a_0 = s_0$ *and* $a_n = s_n - s_{n-1}$, $n \geqslant 1$. *Then* A *accelerates the convergence of* s *iff* $\Delta(a_{n+1}/a_n) = o(1)$.

This theorem shows in particular that the condition

$$a_{n+1}/a_n - r = o(1), \qquad |r| < 1,$$

is not necessary for A to accelerate the convergence of s. However, in the past, Aitken's δ^2-process was usually not applied unless this condition was satisfied. More important, this theorem characterizes in a simple manner the largest class of complex sequences satisfying (4) which are accelerated by A.

I have singled out Aitken's δ^2-process at least partially because it serves as a prototype when studying regularity and, in particular, accelerativity of nonlinear transformations, and since it has appeared so frequently in the literature since 1926.

It also leads us in a natural manner to the Shanks-Schmidt e_k transformation, $k \geqslant 0$, which is an important generalization of Aitken's δ^2-process for $k \geqslant 2$, and for which $e_1 = A$. Formula (3) may be written as the quotient of two determinants of order 2, and it is this form in which e_k generalizes A as a quotient of two determinants of order $k + 1$.

The author rightly refers to e_k as the Schmidt transformation since Schmidt (1941) used the method to solve systems of linear equations by iteration. But his paper was neglected for some time, and the rediscovery of the method is due to Daniel Shanks, who resurrected the algorithm and discussed its remarkable properties in a lengthy paper (1955). There he treated not only regularity and accelerativity, but also analytic continuation in the complex plane.

A very important and useful generalization of the Schmidt transformation e_k to sequences in a Banach space B is the Brezinski-Havie (BH) protocol E_k found in Chapter 10. Its deltoid computation $[E_k(s)]_n = s(n, k)$ is due to Havie (1979), while its representation by a ratio of determinants is due to Brezinski. It is important to note that the denominator of this ratio, being a scalar, does not require anything in the way of invertibility from elements of B.

For scalar sequences, E_k is undoubtedly the most elegant and flexible computational procedure yet discovered for producing sequence transformations. The flexibility of this algorithm lies in the appropriate choices of the elements appearing in the determinants of the ratio mentioned above. Generally speaking, the choices are made with a foreknowledge of the kinds of sequences one wishes to accelerate.

A few general comments are due at this time. This book brings together a large amount of recent results on regularity and accelerativity of transformations of sequences and integrals, and only a small, but fundamental, portion of the material in the book has been considered here. A considerable amount of notation is used, but this seems to be necessary because of the broad coverage of the subject matter.

The present state of the art for many topics in numerical analysis involving sequence transformations is frequently indicated, along with open research problems. Consequently this book is a must for researchers interested in regularity and accelerativity. My own preference of research has been considerably broadened while reviewing it.

The reader of this book should first acquaint himself with the notation. For example, when dealing with real sequences x and y, notations such as "$x_n \leqslant . \; y_n$" ($x_n \leqslant y_n$ for all sufficiently large n) and "$x_n \leqslant : \; y_n$" ($x_n \leqslant y_n$ for infinitely many values of n), first used in Tucker (1963) and later in Tucker (1967, 1969), could make the subject matter completely opaque if not understood.

The bibliography is excellent, but a point of increase of a function is left undefined, as well as the standard notation "$\mathscr{T}(X, Y)$" denoting the family of all functions from X to Y. Proofs are included only if they are either short or conceptually important for the discussion at hand. However, again because of the broad coverage of the subject matter, I feel that this is quite acceptable.

As promised, we now return to the rather paradoxical situation, occurring not infrequently in the general literature, where the convergence of a real sequence s is initially shown to be accelerated by a transformation T, and then the first few terms of s and $t = T[s]$ are exhibited to show the improved approximation to the limit s^* of s by t_n over s_n for small values of n; but only by also displaying a "predetermined" accurate approximation of s^*.

Since the value of s^* is usually unknown, what is really needed in the case of a convergent real sequence s are "sharp" upper and lower bounds on at least the beginning terms of the sequence $\{s^* - s_n\}$. It is preferable to have such bounds on all of the terms of this sequence in the form $b_n \leqslant s^* - s_n \leqslant c_n$, $n \geqslant 0$, along with the added conditions that $q_n = (c_n - b_n)/|s^* - s_n| \to 0$ and q_n be near zero even for small values of n. Finding such bounds may be very difficult to do. The book under review might well have included results from Pinsky (1978) and Johnsonbaugh (1979) which consider this problem for alternating series.

With the conditions in the preceding paragraph being met, we are able to obtain two desired objectives when dealing with a slowly convergent real sequence s. Namely, furnishing (a) accelerative transformations T and (b) accompanying sharp upper bounds for the terms of the sequence $\{|s^* - t_n|\}$ which may be used for "small" values of n in numerical applications.

A procedure to obtain these objectives is, of course, to choose any sequence $\{h_n\}$ such that $b_n \leqslant h_n \leqslant c_n$, $n \geqslant 0$, and define the transformation T by setting

$$[T(s)]_n = t_n = s_n + h_n, \qquad n \geqslant 0.$$

We then have

$$|s^* - t_n| \leqslant c_n - b_n, \qquad n \geqslant 0,$$

and

$$|s^* - t_n|/|s^* - s_n| \leqslant q_n \to 0,$$

i.e., t accelerates the convergence of s. With "all" terms of the sequence q sufficiently near to zero, we have obtained the desired objectives (a) and (b).

Kummer (1837) showed how to carry out that part of the above procedure dealing with the beginning terms of the sequence $\{s_n\}$ of partial sums of a positive term series $a_0 + a_1 + a_2 + \cdots$. A general theory for the convergence of alternating series (i.e., if $a_{n+1}/a_n < 0$, $n \geqslant 0$) is given in Tucker (1963) and Tucker (in preparation). In principle, this general theory can be used to carry out the above procedure in its entirety for the sequence of partial sums of "any" convergent alternating series.

In particular, the error analysis resulting from this general theory for alternating series can be used to derive Aitken's δ^2-process, and also to explain its success as concerns the previously cited objectives (a) and (b) when applied to the partial sums of many slowly convergent alternating series. In fact, it was my attempt to explain the success of this process when applied to such a series which initiated my research on sequence transformations and led me to a general theory of convergence for alternating series.

A rather isolated theorem is proven in Calabrese (1962) which gives upper bounds for the absolute values of the sequence of remainders $s^* - s_n$ of an alternating series which are an improvement over the usual bounds found in most calculus textbooks (see, for example, [**12**, p. 587]). But there appears to be an error in the proof which I believe has never been noted until now. In particular, the erroneous property is used that in "every" (convergent) alternating series the limit (i.e., the sum of the series) must lie between any two successive (partial) sums. Fortunately, it can be easily shown under the hypotheses of the theorem in question that this property does hold.

Valid proofs of the Calabrese result appear in both Pinsky (1978) and Johnsonbaugh (1979). These two papers, the latter in particular, also contain much improved results for approximating the sums of slowly convergent alternating series. However, most of the analysis in each of these three papers depends upon the hypothesis that the sequence $\{|a_n|\}$ is decreasing, along with other restrictions. Eventually some of these neglected results should appear in elementary calculus textbooks. Boas (1978) obtains similar results, but primarily for series of positive terms.

The general theory of alternating series previously mentioned above makes no restrictions other than that the alternating series be convergent. This general theory has yet to be fully exploited. As far as I know, there is no general theory for real series which is analogous to this general theory for alternating series.

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Linear turning point theory, by Wolfgang Wasow, Applied Mathematical Sciences, vol. 54, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1985, ix + 246 pp., $38.00. ISBN 0-387-96046-5

The transitions of solutions of a linear differential equation from oscillatory to exponentially growing or exponentially decaying behavior as the independent variable, for example, changes sign are phenomena of interest to physicists and other scientists, primarily in the past sixty years, and continuing even today. The simplest equation exhibiting such behavior is Airy's equation

$$\text{(A)} \qquad y'' + xy = 0,$$

obtained from his study of the rainbow [2]. If we set $x = [\psi(0)\varepsilon^{-2}]^{1/3}t$, then the rainbow equation (A) becomes

$$\text{(A*)} \qquad \varepsilon^2 d^2y/dt^2 + t\psi(0)y = 0,$$

which might well be expected to have solutions close to solutions of the equation

$$\text{(A\#)} \qquad \varepsilon^2 d^2y/dt^2 + t\psi(t)y = 0 \qquad (\psi(0) \neq 0).$$

This idea occured to R. Gans in 1915 [6] in his investigations of total reflection in physical—as opposed to geometrical—optics. The point $t = 0$ is called a (simple) turning point, one where solutions of (A#) change from oscillatory to exponential behavior. An obvious mathematical question, only answered much later by R. E. Langer [8, 9] and others, is whether one can find changes of variables in (A) such that (A#) becomes (A*) with $\psi(0) = 1$ and with a small error term included. If the transformation from (A#) to A(*) is exact, it turns out that often it is but a formal power series in ε with coefficients holomorphic in the complex variable x, i.e., an asymptotic series which converges only for