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COUNTEREXAMPLES IN THE THEORY OF NONSELFADJOINT OPERATOR ALGEBRAS

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In this note we announce the answers to several questions which involve nonselfadjoint operator algebras. Detailed proofs will appear elsewhere.

We use the following notation. $\mathcal{H}$ is a separable Hilbert space, $\mathcal{B}(\mathcal{H})$ is the algebra of bounded linear operators on $\mathcal{H}$, and $\mathcal{B}_1(\mathcal{H})$ is the ideal of trace class operators on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, $\{T\}'$ is the commutant of T and $\{T\}''$ is the double commutant of T.

$\mathcal{B}(\mathcal{H})$ is the dual of $\mathcal{B}_1(\mathcal{H})$ (see [**2**]) so that $\mathcal{B}(\mathcal{H})$ has a weak * topology. $\mathcal{A}(T)$ denotes the smallest weak * closed algebra containing T and I, while $\mathcal{W}(T)$ is the smallest weak operator closed algebra containing T and I. $\operatorname{Lat} T$ is the lattice of (closed) invariant subspaces of T, and $\operatorname{Alg}\operatorname{Lat} T = \{B \in \mathcal{B}(\mathcal{H})\colon \operatorname{Lat} T \subset \operatorname{Lat} B\}$. It is elementary that $\mathcal{A}(T) \subset \mathcal{W}(T) \subset \{T\}'' \subset \{T\}'$, that $\mathcal{W}(T) \subset \operatorname{Alg}\operatorname{Lat} T$, and that all of these sets except $\mathcal{A}(T)$ are weakly closed algebras. Further, T is said to be reflexive if $\mathcal{W}(T) = \operatorname{Alg}\operatorname{Lat} T$.

We will consider the following questions.

QUESTION 1. Does $\mathcal{W}(T) = \{T\}' \cap \operatorname{Alg}\operatorname{Lat} T$, $\forall T \in \mathcal{B}(\mathcal{H})$?

QUESTION 2. Does $\mathcal{W}(T) = \{T\}'' \cap \operatorname{Alg}\operatorname{Lat} T$, $\forall T \in \mathcal{B}(\mathcal{H})$?

QUESTION 3. Must $T^{(n)}$ be reflexive, $\forall T \in \mathcal{B}(\mathcal{H})$ and $\forall n > 1$? (Here $T^{(n)}$ denotes the direct sum of n copies of T.)

QUESTION 4. If T_1 and T_2 are reflexive operators, must $T_1 \oplus T_2$ be reflexive?

QUESTION 5. Does $\mathcal{A}(T) = \mathcal{W}(T)$, $\forall T \in \mathcal{B}(\mathcal{H})$?

QUESTION 6. Does $\mathcal{W}(T)$ have a separating vector, $\forall T \subset \mathcal{B}(\mathcal{H})$?

Before stating the last question, we need some additional notation. Since $\mathcal{W}(T)$ is weak * closed in $\mathcal{B}(\mathcal{H})$, $\mathcal{W}(T)$ is a dual space, with predual $\mathcal{W}(T)_* = \mathcal{B}_1(\mathcal{H})/\mathcal{W}(T)_\perp$. Here $\mathcal{W}(T)_\perp$ denotes the preannihilator of $\mathcal{W}(T)$. For each n, let $F_n \subset \mathcal{B}_1(\mathcal{H})$ denote the set of operators of rank $\leq n$.

QUESTION 7. Is $F_1/\mathcal{W}(T)_\perp$ dense in $\mathcal{W}(T)_*$, $\forall T \subset \mathcal{B}(\mathcal{H})$?

Some remarks regarding these questions are in order. There are some relations among the questions. For $n = 1, 2$, or 6, an affirmative answer to Question n implies an affirmative answer to Question $n+1$.

Question 1 was raised independently by D. Sarason and P. Rosenthal (see [**6**, p. 195] and [**7**]). Rosenthal also asked Question 2 in [**7**]. In [**4**], J. Deddens listed several open questions, including Questions 3 and 4, concerning reflexive operators.

Question 5 has been raised by many people. The question appears in [**2**]. In [**8**], D. Westwood gave an example of an operator T so that $\mathcal{A}(T) = \mathcal{W}(T)$ but so that the weak and weak * topologies are different on $\mathcal{A}(T)$.

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Questions 6 and 7 were raised by D. Larson in a private communication. The motivation for the questions arose from the following. There has been intense research activity (see [**1**, **2**, and **3**], e.g.) on operators T such that every weak * continuous linear functional on $\mathcal{W}(T)$ is represented by a rank one operator. (Thus T satisfies $\mathcal{W}(T)_* = F_1/\mathcal{W}(T)_\perp$.) There are operators T which do not have this property (see [**5** and **1**]), but for these operators T, $F_1/\mathcal{W}(T)_\perp$ is dense in $\mathcal{W}(T)_*$.

We have been able to show that all seven of these questions have a negative answer. The key to the construction of the counterexamples is the following theorem.

THEOREM. *Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces with* $\dim \mathcal{K} = \infty$. *Let $\mathcal{S}$ be a weakly closed subspace of $\mathcal{B}(\mathcal{H})$. Then there is an operator $T \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K} \oplus \mathcal{H})$ of form*

$$T = \begin{pmatrix} 0 & P & 0 \\ 0 & W & Q \\ 0 & 0 & 0 \end{pmatrix}$$

so that $\mathcal{W}(T)$ splits as an independent direct sum: $\mathcal{W}(T) = \mathcal{B}(T) \dotplus \tilde{\mathcal{S}}$, where $\tilde{\mathcal{S}} = \{A \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K} \oplus \mathcal{H})\colon A_{1,3} \in \mathcal{S}$ and $A_{i,j} = 0$ if $(i,j) \neq (1,3)\}$ and $\mathcal{B}(T) = \{A \in \mathcal{W}(T)\colon A_{1,3} = 0\}$.

We now indicate how this theorem settles Question 1. Let $\mathcal{H} = \mathbf{C}^2$ and let $\mathcal{S}$ be the set of trace zero operators on $\mathcal{H}$. Then $\mathcal{S}$ is a transitive subspace of $\mathcal{B}(\mathcal{H})$. This means (see [**1**]) that $\mathcal{S}x = \mathcal{H}$ for all $x \in \mathcal{H}$, $x \neq 0$. Construct T as in the theorem, so that $\mathcal{W}(T) = \mathcal{B}(T) \dotplus \tilde{\mathcal{S}}$. Now every $A \in \widetilde{\mathcal{B}(\mathcal{H})}$ is nonzero only in its $(1,3)$ entry, so $AT = TA = 0$ and $A \in \{T\}'$. Also, using transitivity of $\mathcal{S}$, it is easy to see that $A \in \operatorname{Alg}\operatorname{Lat} T$. $\mathcal{S}$ is a proper subspace, so $\widetilde{\mathcal{B}(\mathcal{H})}$ is not contained in $\mathcal{W}(T)$ and we have a counterexample. We note that this example was motivated in part by the excellent survey of some finite dimension results which appears in the beginning of the paper [**1**] of E. Azoff.

It is easy to check that choosing $\mathcal{S} = \mathcal{B}(\mathcal{H})$ in the theorem yields a counterexample to Questions 6 and 7. Some additional information on the structure of the subspace $\mathcal{B}(T)$ is required in order to give examples settling the remaining questions.

We now outline the proof of the theorem. We identify $\mathcal{K}$ with $\bigoplus_1^\infty \mathcal{H}$. In the matrix for T let P be the isometry of $\mathcal{H}$ into $\mathcal{K}$ with matrix $(I\ 0\ 0 \cdots)$. Let W be a backward operator weighted shift with weight sequence $(w_n I)$ to be specified later. Thus W has matrix $(W_{i,j})$ where $W_{n,n+1} = w_n I$, $n \geq 1$, and all other entries $= 0$. Let $\mathcal{C}$ be a countable weakly dense set in the unit ball of $\mathcal{S}$. Let (Q_n) be a sequence in $\mathcal{C}$ so that each $C \in \mathcal{C}$ appears infinitely often in (Q_n). Since Q is to be an operator from $\mathcal{K}$ to $\mathcal{H}$, we think of Q as an operator matrix with one column. Let the nth entry of this column be $b_n Q_n$. Here we assume $b_n \neq 0$ $\forall n$ and that $(b_n) \in l^2$. This insures that Q is bounded.

If $n \geq 1$, then

$$T^{n+1} = \begin{pmatrix} 0 & PW^n & PW^{n-1}Q \\ 0 & W^{n+1} & W^nQ \\ 0 & 0 & 0 \end{pmatrix}.$$

Now $PW^{n-1}Q = \lambda_n Q_n$, where $\lambda_n = w_1 w_2 \cdots w_{n-1} b_n$. Consider the sequence $((1/\lambda_n)T^{n+1})$. If the weights w_n are chosen to go to zero sufficiently quickly, then all matrix entries of $(1/\lambda_n)T^{n+1}$ except for the $(1,3)$ entry go to zero with n. It follows that $\tilde{S} \subset \mathcal{W}(T)$.

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REFERENCES

1. E. Azoff, *On finite rank operators and preannihilators*, preprint.

2. H. Bèrkovici, C. Foias, and C. Pearcy, *Invariant subspaces, dilation theory, and dual algebras*, CBMS Regional Conf. Ser. in Math., no. 56, Amer. Math. Soc., Providence, R. I., 1985.

3. S. Brown, *Some invariant subspaces for subnormal operators*, Integral Equations and Operator Theory **1** (1978), 310–333.

4. J. Deddens, *Reflexive operators*, Indiana Univ. Math. J. **20** (1971), 887–889.

5. D. Hadwin and E. Nordgren, *Subalgebras of reflexive algebras*, J. Operator Theory **7** (1982), 3–23.

6. H. Radjavi and P. Rosenthal, *Invariant subspaces*, Springer-Verlag, New York, 1973.

7. P. Rosenthal, *Problems in invariant subspaces and operator algebras*, Colloq. Math. Soc. János Bolyai, vol. 5, North-Holland, Amsterdam, 1970.

8. D. Westwood, *Weak operator and weak * topologies on singly generated algebras*, J. Operator Theory (submitted).

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