FREUD'S CONJECTURE FOR EXPONENTIAL WEIGHTS

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1. Freud's Conjecture. Let $W(x)$ be nonnegative on $\mathbf{R}$, positive on a set of positive Lebesgue measure, and let $W^2(x)$ have all moments finite. Then we call W a *weight function*. Associated with $W^2(x)$ are the orthonormal polynomials

$$p_n(W^2; x) := \gamma_n(W^2)x^n + \cdots, \qquad n = 0, 1, 2, 3, \ldots,$$

satisfying

$$\int_{-\infty}^{\infty} p_m(W^2; x)p_n(W^2; x)W^2(x)\,dx = \delta_{mn}.$$

While asymptotics for the ratio γ_{n-1}/γ_n, $n \to \infty$, are classical for weights on $[-1, 1]$, only recently have analogous results been considered for the more difficult problem of weights on $\mathbf{R}$. In 1974, G. Freud [2] conjectured that if $W(x) = W_{\alpha,\rho}(x)$, where

$$W_{\alpha,\rho}(x) := |x|^{\rho/2}\exp(-|x|^\alpha), \qquad x \in \mathbf{R},\ \alpha > 0,\ \rho > -1,$$

then

$$\lim_{n\to\infty} n^{-1/\alpha}\gamma_{n-1}(W_{\alpha,\rho}^2)/\gamma_n(W_{\alpha,\rho}^2)$$

exists. He expressed the value that the limit should take in terms of gamma functions, and proved his conjecture for $\alpha = 2, 4, 6$. Recently, Al. Magnus [8] proved the conjecture for $\rho > -1$ and α a positive even integer, and subsequently [9] for weights of the form $\exp(-P(x))$, where $P(x)$ is a polynomial of even degree with positive leading coefficient. Máté, Nevai, and Zaslavsky [11] have sharpened Magnus' result to an asymptotic expansion. Several applications of Freud's Conjecture are discussed by Nevai [16], and related physical applications have been considered by Bessis, Itzykson, and Zuber [1] and in [17].

The purpose of this paper is to announce a proof of Freud's Conjecture for a general class of weights that includes $W_{\alpha,\rho}(x)$ for all $\alpha > 0$, $\rho > -1$. In describing the analogue of the conjecture for general weights, a crucial role is played by the number $a_n = a_n(W)$, introduced by Mhaskar and Saff in [13, 14]. Let $W(x) = \exp(-Q(x))$, where $Q(x)$ is even, continuous in $\mathbf{R}$, and

differentiable for $x > 0$, while $xQ'(x)$ remains bounded as $x \to 0$. Whenever uniquely defined, $a_n = a_n(W)$ is taken to be the positive root of the equation

$$(1.1) \qquad n = \frac{2}{\pi} \int_0^1 \frac{a_n x Q'(a_n x)\, dx}{\sqrt{1 - x^2}}.$$

Under certain conditions on Q, it is shown in [14] that for every polynomial $P(x)$ of degree at most n,

$$(1.2) \qquad \|PW\|_{L_\infty(\mathbf{R})} = \|PW\|_{L_\infty[-a_n, a_n]}$$

and that a_n is asymptotically best possible.

The set of weights for which we can prove Freud's Conjecture includes the following class.

DEFINITION 1.1. Let $W(x) = \exp(-Q(x))$, where $Q(x)$ is even and continuous and $Q'''(x)$ exists for $x > 0$, while $xQ'(x)$ is bounded as $x \to 0+$, and for some $C > 0$ and $\alpha > 0$,

$$(1.3) \qquad Q'(x) > 0, \qquad x \text{ large enough},$$

$$(1.4) \qquad x^2 |Q'''(x)| / Q'(x) \le C, \qquad x \text{ large enough},$$

and

$$(1.5) \qquad \lim_{x \to \infty} \left(1 + xQ''(x)/Q'(x) \right) = \alpha.$$

Then we shall call W a very smooth Freud weight of order α and write $W \in \text{VSF}(\alpha)$.

The conditions on Q may be slightly weakened, but we retain the above formulation for simplicity. These conditions arise in the construction of an even entire function with nonnegative Maclaurin series coefficients that behaves like $W^{-1}(x)$ on $\mathbf{R}$ (Lubinsky [4, 5]), and in ensuring that the zero distribution of extremal polynomials associated with W behaves asymptotically like the Ullman distribution [6, 14]. In particular, if $Q(x) = |x|^\alpha (\log(2 + x^2))^\beta$, $\alpha > 0$, $\beta \in \mathbf{R}$, then $W(x) = \exp(-Q(x)) \in \text{VSF}(\alpha)$. We note that if $W \in \text{VSF}(\alpha)$, then a_n is uniquely defined by (1.1) for n large enough.

THEOREM 1.2. Let $W \in \text{VSF}(\alpha)$ for some $\alpha > 0$ and let $\rho > -1$. If $a_n := a_n(W)$ is defined by (1.1) and $\gamma_n := \gamma_n(|x|^\rho W^2)$ are the leading coefficients of the orthonormal polynomials associated with $|x|^\rho W^2$, then

$$(1.6) \qquad \lim_{n \to \infty} a_n^{-1} \gamma_{n-1}/\gamma_n = 1/2.$$

In the special case $W(x) = \exp(-|x|^\alpha)$, $\alpha > 0$, (1.6) may be rewritten in the form

$$(1.7) \qquad \lim_{n \to \infty} n^{-1/\alpha} \gamma_{n-1}(|x|^\rho W_\alpha^2)/\gamma_n(|x|^\rho W_\alpha^2) = \beta_\alpha/2,$$

where

$$(1.8) \qquad \beta_\alpha := \left\{ \frac{\Gamma(\alpha)}{2^{\alpha-2}\Gamma(\alpha/2)^2} \right\}^{-1/\alpha}.$$

We shall also prove in [**7**] the analogue of Freud's Conjecture for noneven weights such as $|x|^\rho W^2 e^{-P}$, where $\rho > -1$, $W \in \text{VSF}(\alpha)$ for some $\alpha > 0$, and P is a polynomial of degree less than α.

NOTE ADDED IN PROOF. Asymptotics for $\gamma_n(W)$ and its L_p analogue $(1 \le p \le \infty)$, rather than just $\gamma_{n-1}(W)/\gamma_n(W)$, have also been obtained for weights such as $\exp(-|x|^\alpha)$, $\alpha > 0$, but for a less general class than in Theorem 1.2.

2. Outline of the proof. We shall briefly discuss the main threads of the proof of Theorem 1.2. Note first that one equivalent formulation of (1.2) is

$$(2.1) \qquad \|P_n(x)W(a_n x)\|_{L_\infty(\mathbf{R})} = \|P_n(x)W(a_n x)\|_{L_\infty[-1,1]},$$

for each polynomial $P_n(x)$ of degree at most n. Under mild conditions on W, results in [**13**, **14**] imply that any sequence of weighted polynomials $\{P_n(x)W(a_n x)\}_{n=1}^\infty$ uniformly bounded in $\mathbf{R}$ will converge to zero outside $[-1,1]$. Concerning the behavior of such sequences in $[-1,1]$, the following result is proved in Lubinsky and Saff [**6**].

THEOREM 2.1. *Let* $W \in \text{VSF}(\alpha)$ *for some* $\alpha > 0$ *and* $a_n := a_n(W)$ *be defined by* (1.1). *Let* $g \in L_p(\mathbf{R})$, $0 < p < \infty$. *Then there exist polynomials* $P_n(x)$ *of respective degrees at most* n, $n = 1, 2, 3, \ldots$, *such that*

$$(2.2) \qquad \lim_{n \to \infty} \|g(x) - P_n(x)W(a_n x)\|_{L_p(\mathbf{R})} = 0$$

if and only if $g(x) = 0$ *for almost all* $|x| \ge 1$.

There is also an L_∞ analogue of Theorem 2.1 which appears in [**6**]. One corollary to Theorem 2.1 is the solution to an L_p analogue of a problem of Saff [**18**, p. 252], concerning approximation by weighted polynomials of the form $P_n(x) \exp(-n|x|^\alpha)$:

COROLLARY 2.2. *Let* $\alpha > 0$. *Let* $0 < p < \infty$ *and* $g \in L_p(\mathbf{R})$. *Then there exist polynomials* $P_n(x)$ *of respective degrees at most* n, $n = 1, 2, 3, \ldots$, *such that*

$$\lim_{n \to \infty} \|g(x) - \exp(-n|x|^\alpha)P_n(x)\|_{L_p(\mathbf{R})} = 0,$$

if and only if $g(x) = 0$ *for almost all* $|x| \ge \beta_\alpha$.

A second ingredient in the proof of Theorem 1.2 is the following sufficient condition for Freud's Conjecture for even weights [**3**]. An analogue for noneven weights will also appear in [**3**]. Its proof relies entirely on methods and results developed by Máté, Nevai, and Totik [**10**] in their proof of Rakhmanov's theorem and extensions thereof.

THEOREM 2.3. *Let* W *be a weight function. Suppose there exists an increasing sequence of positive numbers* $\{c_n\}_{n=1}^\infty$ *and a decreasing sequence of positive numbers* $\{\delta_n\}_{n=1}^\infty$ *with*

$$(2.3) \qquad \lim_{n \to \infty} \delta_n = 0$$

such that for $n = 1, 2, 3, \ldots$, and every polynomial $P(x)$ of degree at most n,

$$(2.4) \qquad \|PW\|_{L_2(\mathbf{R})} \leq (1 + \delta_n)\|PW\|_{L_2[-c_n, c_n]}.$$

Further, suppose there exist polynomials $S_n(x)$ of respective degrees at most $n - 2$, $n = 2, 3, \ldots$, such that for $p = 1/2$ and $p = 2$,

$$(2.5) \qquad \lim_{n \to \infty} \frac{1}{\pi} \int_{-1}^{1} |S_n(x)W(c_n x)(1 - x^2)^{1/4}|^p \frac{dx}{\sqrt{1 - x^2}} = 1.$$

Then

$$\lim_{n \to \infty} c_n^{-1} \gamma_{n-1}(W^2)/\gamma_n(W^2) = 1/2.$$

It is not difficult to see that if c_n in (2.5) is replaced by a_n, then (2.5) is an essentially weaker form of (2.2) with $p = \infty$. The final ingredient in the proof is to establish L_2 inequalities like those in (2.4), with c_n replaced by $a_n(1 + \varepsilon_n)$, where $\{\varepsilon_n\}_{n=1}^{\infty}$ is a sequence of positive numbers with

$$\lim_{n \to \infty} \varepsilon_n = 0.$$

Although such inequalities are implicit in results of Mhaskar and Saff [**12**, **14**, **15**], they have not been formally stated in the literature. One consequence of results of Mhaskar and Saff [**12**, **15**] is the following:

THEOREM 2.4. *Let $W \in \mathrm{VSF}(\alpha)$ for some $\alpha > 0$. Let $a_n := a_n(W)$ be defined by (1.1) for n large enough. Let $0 < p \leq \infty$ and $\delta > 0$. Then there exists $C_1, C_2 > 0$ such that for $n = 1, 2, 3, \ldots$, and all polynomials P of degree at most n,*

$$\|P(x)W(x)\|_{L_p(\mathbf{R})} \leq (1 + C_1 e^{-C_2 n})\|P(x)W(x)\|_{L_p(-a_n(1+\delta), a_n(1+\delta))}.$$

This is stated and proved in [**6**]. The precise inequalities required for application of Theorem 2.3, as well as full details of the proof of Theorem 1.2, will appear in [**7**].

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