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DIRECT IMAGES OF HERMITIAN HOLOMORPHIC BUNDLES

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Introduction. We introduce higher analogues of analytic torsion, which are form valued. Using this construction we obtain, in the case of the projection map for a product, a Grothendieck-Riemann-Roch theorem for hermitian holomorphic vector bundles which is an equality between differential forms. This is related to work of Quillen [**6**] and of Bismut and Freed [**1**].

I. A Grothendieck group.

I.1. Let X be a complex manifold. For any $p \in \mathbf{N}$ let $A^{p,p}(X)$ be the space of real (p,p) forms over X. Let $A(X) = \bigoplus_{p\geq 0} A^{p,p}(X)$, and $\tilde{A}(X) = A(X)/(\mathrm{Im}(\partial) + \mathrm{Im}(\overline{\partial}))$, where $d = \partial + \overline{\partial}$ is the standard decomposition of the exterior derivative on X.

I.2. An *hermitian holomorphic bundle* (or h.h. bundle) on X is a pair $\overline{E} = (E, h)$, consisting of a finite-dimensional complex holomorphic vector bundle E over X and a smooth hermitian scalar product h on E. Given $\overline{E}$, let ∇ be the unique connection on E which is both compatible with its complex structure and unitary for h, as in [**2**]. The closed form $\mathrm{ch}(\overline{E}) = \mathrm{Tr}(\exp((i/2\pi)\nabla^2)$ in $A(X)$ represents the Chern character of E.

I.3. Let $\hat{K}_0(X)$ be the abelian group generated by pairs $(\overline{E}, \eta)$ where E is an h.h. bundle over X and $\eta \in \tilde{A}(X)$, with the following relations. Let

$$\overline{\mathcal{E}}: 0 \to \overline{S} \to \overline{E} \to \overline{Q} \to 0$$

be any exact sequence of holomorphic bundles over X, endowed with arbitrary metrics, and $\eta', \eta'' \in \tilde{A}(X)$. We impose the relation $(\overline{S}; \eta') + (\overline{Q}; \eta'') = (\overline{E}; \eta' + \eta'' - \widetilde{\mathrm{ch}}(\overline{\mathcal{E}}))$, where $\widetilde{\mathrm{ch}}(\overline{\mathcal{E}}) \in \tilde{A}(X)$ is the solution to the equation

$$(1/\pi i)\partial\overline{\partial}\widetilde{\mathrm{ch}}(\overline{\mathcal{E}}) = \mathrm{ch}(\overline{S}) + \mathrm{ch}(\overline{Q}) - \mathrm{ch}(\overline{E})$$

introduced by Bott and Chern in [**2**].

I.4. The following construction of $\widetilde{\mathrm{ch}}$ is used in the proofs of the results below. Let $\mathcal{O}(1)$ be the tautological line bundle on the complex projective line $\mathbf{P}^1$, and let z be the parameter on the affine line $\mathbf{A}^1 \subset \mathbf{P}^1$. If $\sigma: \mathcal{O} \to \mathcal{O}(1)$ is the section vanishing at infinity, let $s = \mathrm{Id} \otimes \sigma$ be the induced map $S \to S(1)$ on $X \times \mathbf{P}^1$. If $i: S \to E$ is the inclusion in $\overline{\mathcal{E}}$ above, let $F = (S(1) \oplus E)/S$ be the vector bundle which is the cokernel of $s \oplus i$. If $i_p: X \times \{p\} \to X \times \mathbf{P}^1$ for $p = 0, \infty$ are the natural inclusions, then $i_0^* F \simeq E$ while $i_\infty^* F \simeq S \oplus Q$. We may choose a metric on F so that these maps are isometries. Then, in $\tilde{A}(X)$:

$$\widetilde{\mathrm{ch}}(\overline{\mathcal{E}}) = \int_z \mathrm{ch}(\overline{F}) \log |z|.$$

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I.5. Define a homomorphism ch: $\widehat{K_0}(X) \to A(X)$ by setting

$$\mathrm{ch}(\overline{E};\eta) = \mathrm{ch}(\overline{E}) - (1/\pi i)\partial\overline{\partial}\eta.$$

I.6. When X is algebraic over a ring R contained in $\mathbf{C}$ one can require, when defining $\widehat{K_0}(X)$, that the bundles be algebraic over R (cf. [**3**]). The results below remain true in that case.

II. Higher analytic torsion.

II.1. Let X be a complex manifold, Y a compact Kähler manifold and $f: X \times Y \to X$ the projection onto the first factor.

Let $\overline{E}$ be an h.h. bundle over $X \times Y$. For any point x in X, we get an h.h. bundle $\overline{E}_x$ on Y by restricting $\overline{E}$ to the fibre $f^{-1}(x) \simeq Y$. In this section we assume that

(A) *For all x in X and all $k > 0$, the cohomology group $H^k(Y, E_x)$ vanishes.*

Under this assumption, we shall define a form $T(f, \overline{E})$ in $\tilde{A}(X)$, called the *higher analytic torsion* of E (relative to f).

II.2. From (A) we know that there is a vector bundle $f_*(E)$ on X the fibre of which at any $x \in X$ is equal to $H^0(Y, E_x)$. For any $q \in \mathbf{N}$ let $T^*(Y)^{0,q}$ be $(0, q)$ part of the qth exterior power of the complexified cotangent bundle to Y. Let D^q be the smooth infinite-dimensional bundle over X, with fiber at $x \in X$ equal to the space of smooth sections over Y of the bundle $E_x \otimes T^*(Y)^{0,q}$. The Dolbeault resolution of E_x over Y, as $x \in X$ varies, gives rise to an acyclic complex of bundles on X:

$$0 \to f_*(E) \to D^0 \to D^1 \to D^2 \to \cdots.$$

II.3. The choice of metrics on Y and E determines a metric on $E_x \otimes T^*(Y)^{0,q}$ and an L^2 scalar product on D^q, $q \geq 0$. Let $f_*(h)$ be the hermitian metric on $f_*(E)$ induced from D^0, and ∇ the associated connection (I.2). We also define a connection ∇ on D^q, $q \geq 0$, as follows. Giving a section s of D^q over X is equivalent to giving a section of the bundle $E \otimes \pi^*(T^*(Y)^{0,q})$ over $X \times Y$, where $\pi: X \times Y \to Y$ is the second projection. The latter bundle has a metric and therefore a connection $\tilde{\nabla}$. If v is a tangent vector to X at the point $x = f(x, y)$, let $\tilde{v}$ be the horizontal tangent vector to $X \times Y$ at (x, y) such that $f(\tilde{v}) = v$. We define $\nabla_v(s)(x, y)$ to be $\tilde{\nabla}_{\tilde{v}}(s)(x, y)$.

II.4. We write H^+ for the pull-back from X to $X \times \mathbf{C}^*$ of $\bigoplus_{i \geq 0} D^{2i}$, and H^- for the pull back of $\bigoplus_{i \geq 0} D^{2i+1} \oplus f_*(E)$. These bundles admit the connection

$$\nabla_z = \nabla + \frac{\partial}{\partial z}\, dz + \frac{\partial}{\partial \overline{z}}\, d\overline{z},$$

where z is the coordinate in $\mathbf{C}^*$. Let $\overline{\partial}_Y^*$ (resp. j^*) be the adjoint of $\overline{\partial}_Y$ (resp. j), and $L_z: H^+ \oplus H^- \to H^+ \oplus H^-$ be the odd endomorphism $i(z\overline{\partial}_Y \oplus \overline{z}\overline{\partial}_Y^* \oplus zj \oplus \overline{z}j^*)$. Following Quillen [**5**], we give the super vector bundle $H^+ \oplus H^-$ the superconnection $\nabla_z + L_z$. The operator $\exp(\nabla_z + L_z)^2$ is trace class on $H^+ \oplus H^-$ and its supertrace

$$\omega(z) = \mathrm{tr}_s \exp(\nabla_z + L_z)^2$$

is a form on $X \times \mathbf{C}^*$ (compare [**5** and **1**]). For any real number $r > 0$ one can show that the integral

$$I(r) = \int_{|z|^2 > r} \omega(z) \log |z|$$

converges, belongs to $A(X)$, and admits an asymptotic development (as r goes to zero) with finitely many divergences of type r^{-j-1} and $r^{-j}\log(r)$, $j \in \mathbf{N}$. Let $I(0) \in A(X)$ be the finite part of $I(r)$ as r goes to zero. By definition, $T(f, \overline{E}) \in \tilde{A}(X)$ is the class of the form obtained from $I(0)$ by multiplying its (p,p) component by $(i/2\pi)^{p+1}$, for all $p \geq 0$.

III. A Riemann-Roch Theorem.

III.1. Assume X, Y and $\overline{E}$ are as in §II, and also that Y is projective. Let $f_!(\overline{E}) \in \widehat{K_0}(X)$ be the class of $(f_*(E), f_*(h); T(f, \overline{E}))$. Let $Td(\overline{Y}) \in A(Y)$ denote the closed form, representing the Todd class of Y, determined by the choice of metric on Y, as in I.2. For any $\eta \in \tilde{A}(X \times Y)$ let $f_!(0;\eta) = (0; f_*(\eta\, Td(\overline{Y}))$ in $\widehat{K_0}(X)$, where f_* denotes integration along Y.

III.2. THEOREM 1. (i) *The map* $f_!$ *extends uniquely to a group homomorphism*

$$f_! \colon \widehat{K_0}(X \times Y) \to \widehat{K_0}(X).$$

(ii) *For any* α *in* $\widehat{K_0}(X \times Y)$, *we have an equality in* $A(X)$:

$$\mathrm{ch}(f_!(\alpha)) = f_*(\mathrm{ch}(\alpha) Td(\overline{Y})).$$

IV. The metric on the determinant bundle.

IV.1. Let $\widehat{\mathrm{Pic}}(X)$ be the group of hermitian holomorphic line bundles over X, modulo biholomorphic isometries. There is a morphism

$$\widehat{\det} \colon \widehat{K_0}(X) \to \widehat{\mathrm{Pic}}(X)$$

sending $(E, h : \eta)$ to the maximal exterior power $L = \bigwedge^{\max}(E)$, given the metric $\bigwedge^{\max}(h)\exp(-2\eta^0)$, where $\eta^0 \in C^\infty(X, \mathbf{R})$ is the component of η of degree zero.

IV.2. Let X and Y be as in II.1 and let $\overline{E}$ be an h.h. bundle over $X \times Y$. On the line bundle $\lambda(E) = \det Rf_*(E)$ over X (cf. [**4**]) we define a metric h as follows. For any point $x \in X$, let Δ^q be the Laplace operator $\overline{\partial}_Y \overline{\partial}_Y^* + \overline{\partial}_Y^* \overline{\partial}_Y$ on D_x^q, $q \geq 0$, H_x^q its kernel, and K_x^q the orthogonal complement to H_x^q in D_x^q. When $s \in \mathbf{C}$ has large enough real part, consider the zeta function $\varsigma^q(s) = \mathrm{Trace}((\Delta^q)^{-s}$ on $K_x^q)$. Now $\varsigma^q(s)$ admits a meromorphic continuation to the whole complex plane, which is regular at the origin. Let

$$\tau(x) = \sum_{q \geq 0} (-1)^q [q(\varsigma^q)'(0) - q\gamma\varsigma^q(0) + (1-q)\gamma \dim_{\mathbf{C}} H_x^q]$$

where γ is the Euler constant. The L^2 metric on H_x^q and the canonical isomorphism between $\lambda(E)_x$ and $\bigotimes_{q \geq 0} (\bigwedge^{\max}(H_x^q))^{(-1)^q}$, [**4**], define a metric h_{L^2} on $\lambda(E)_x$. Let $h_Q = h_{L^2}\exp(-\tau(x))$. One can show that the scalar product h_Q on $\lambda(E)_x$ varies smoothly with x; see [**6** and **1**].

IV.3. THEOREM 2. *If* $\overline{E}$ *satisfies* II.1(A), *the class of* $(\lambda(E), h_Q)$ *in* $\widehat{\mathrm{Pic}}(X)$ *is equal to* $\widehat{\det}(f_!(\overline{E}))$.

IV.4. Theorems 1 and 2 imply that the first Chern form of $(\lambda(E), h_Q)$ is the component of degree two in $f_*(\mathrm{ch}(\overline{E})Td(\overline{Y}))$. The proof of Theorem 1 uses the constructions of I and II, and the local index theorem of Bismut [**7**].

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