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THE CYCLIC HOMOLOGY AND K-THEORY OF CURVES

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ABSTRACT. It is now possible to calculate the K-theory of a large class of singular curves over fields of characteristic zero. Roughly speaking, the K-theory of a curve is the K-theory of its (smooth) normalization plus a few shifted copies of the K-theory of the field plus a "nil part." The nil part is a vector space depending only on the analytic type of the singularities, and may be computed locally. We completely compute the nil part for seminormal curves and give a conjectural calculation in general which depends upon cyclic homology.

Until recently, very little has been known about the higher algebraic K-theory of anything but finite fields. In this note we announce the computability of the K-theory of singular curves in characteristic zero in terms of the K-theory of smooth curves and fields. If the curve is seminormal, we give a complete calculation; otherwise, the calculation depends on the validity of:

CONJECTURE. *Let B be a finite integral extension of a ring A, and let I be the conductor ideal. Assume A contains $\mathbf{Q}$, the rational numbers. Then the map*

$$K_*(A,B,I) \to HC_{*-1}(A,B,I)$$

is an isomorphism, where the right-hand term is double relative cyclic homology taken over the field $\mathbf{Q}$.

This conjecture is known when $B = A/J$ [**OW**]. In the absence of this conjecture, all our results can be interpreted as calculations of the cyclic homology of affine curves. In order to more simply present our results, let us set

$$V_n = \begin{cases} 0 & n = 0, 1, \\ k \oplus \Omega_k^2 \oplus \Omega_k^4 \oplus \cdots \oplus \Omega_k^{n-2} & n \text{ even, } n \geq 2, \\ \Omega_k \oplus \Omega_k^3 \oplus \cdots \oplus \Omega_k^{n-2} & n \text{ odd, } n \geq 3. \end{cases}$$

Here Ω_k^i denotes the ith exterior of the module Ω_k of Kähler differentials of k over $\mathbf{Q}$. As an illustration, we present

CURVE 1 (TWO INTERSECTING LINES). Let k be a field of characteristic zero, and set $A = k[x,y]/(xy)$, $I = (x,y)A$, and $X = \mathrm{Proj}(k[X,Y,Z]/(XY))$.

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Then

$$\begin{aligned} HC_n(A) &= HC_n(k) \oplus V_{n+1} \oplus (I \otimes \Omega_k^n), \\ K_n(A) &= K_n(k) \oplus V_n, \\ K_n(X) &= K_n(A) \oplus K_n(k) \oplus K_n(k). \end{aligned}$$

Fix a field k of characteristic zero and let X be a singular curve over k. We shall compute the K-theory of X by means of a series of reductions. Using analytic isomorphisms [**W1**], we reduce to the case in which X is affine, say $X = \mathrm{Spec}(A)$. If A is not reduced, we can compute the K-theory of A in terms of the K-theory of $A_{\mathrm{red}} = A/N$ (N is the nilradical of A) and the relative groups of N. The latter are computable in terms of cyclic homology; namely, $K_n(A, N) \cong HC_{n-1}(A, N)$. (This isomorphism is due to Goodwillie [**G**].) We thus reduce to the case in which A is reduced.

At this point, we break up the K-theory of A into two pieces: the Karoubi-Villamayor theory and the nil K-theory [**W2**]. The groups $KV_*(A)$ are easily computed in terms of the K-theory of the normalization B of A and the residue fields of A and B at the singular points of A. This procedure is outlined in [**R** and **W3**].

The groups $\mathrm{nil}\, K_*(A)$ are Q-vector spaces, and it is these that we can compute. Since A is reduced and 1-dimensional, $\mathrm{nil}\, K_*(A)$ is the direct sum over the singular primes m of A of the groups $\mathrm{nil}\, K_*(A_m)$. These groups depend only on the analytic type of the singularity, in the sense that if $\hat{A}$ is the m-adic completion of A, then $\mathrm{nil}\, K_*(A_m) \cong \mathrm{nil}\, K_*(\hat{A})$.

To illustrate the nature of the last few reductions, the computation for two intersecting lines allows us to compute

CURVE 2 (NODE). Let $A = k[x, y]/(y^2 = x^2 - x^3)$ and

$$X = \mathrm{Proj}(k[X, Y, Z]/(Y^2 Z = X^2 Z - X^3)).$$

Then $K_n(X) = K_n(A) \oplus K_n(k)$ and

$$K_n(A) = K_n(k) \oplus K_{n+1}(k) \oplus V_n.$$

Similarly, we can calculate the K-theory of any curve whose singularities have linearly independent branches by means of the above reductions and the following special case.

CURVE 3 ($b + 1$ BRANCHES). Let $A = k[x_0, \ldots, x_b]/(x_i x_j = 0, i \neq j)$. Then

$$\begin{aligned} HC_n(A) &= HC_n(k) \oplus \coprod_{c(b,n+1)} k \oplus \coprod_{c(b,n)} \Omega_k \oplus \cdots \oplus \coprod_{c(b,2)} \Omega_k^{n-1} \oplus (I \otimes \Omega_k^n), \\ K_n(A) &= K_n(k) \oplus \coprod_{c(b,n)} k \oplus \prod_{c(b,n-1)} \Omega_k \oplus \cdots \oplus \coprod_{c(b,2)} \Omega_k^{n-2}, \end{aligned}$$

where, letting μ denote the Möbius function,

$$c(b, q) = \sum_{\substack{e \mid q \\ q-e \text{ even}}} \sum_{d \mid e} \frac{\mu(e/d)}{e} [b^d + (-1)^d b].$$

The function $c(b,q)$ counts the number of ways of using the alphabet $\{0,1,\dots,b\}$ to write words of q letters around a circle so that (i) no two adjacent letters are the same and (ii) if rotation by e positions fixes the word then $(-1)^e = (-1)^q$. (For example, the word 012012 is disallowed when $q = 6$.) Since

$c(1,q)$ is 1 if q is even and 0 if q is odd, the case $b = 1$ recovers Curve 1. Note that $c(b,2) = (b^2+b)/2$, $c(b,3) = (b^3-b)/3$, $c(b,4) = (b^4-b^2)/4+(b^2+b)/2$. In general $c(b,q) \sim b^q/q$.

The elements of $K_n(A)$ are easy to describe using "Loday symbols" $\langle\langle f_1,\dots,f_q\rangle\rangle$ [**L**]. For each of the $c(b,q)$ words $i_1,\dots,i_q$ and each element $a_0da_1\wedge\cdots\wedge da_p$ of Ω_k^p we get an element

$$\langle\langle a_0x_{i_1},\dots,x_{i_q}\rangle\rangle \cup \{a_1,\dots,a_p\} \in K_{p+q}(A).$$

Finally, according to [**D**], every seminormal singularity has the analytic type of A^G, where A is the ring of Curve 3 and G is a galois group acting on both k and A in a graded way. Since the galois group acts on the HC and K-groups of A in an obvious way, we have

$$HC_n(A^G) = HC_n(k^G) \oplus \left(\coprod_{c(b,n+1)} k\right)^G \oplus\cdots\oplus (I\otimes\Omega_k^n)^G,$$

$$K_n(A^G) = K_n(k^G) \oplus \left(\coprod_{c(b,n)} k\right)^G \oplus\cdots\oplus \left(\coprod_{c(b,2)} \Omega_k^{n-2}\right)^G.$$

This completes the computation of the K-theory of seminormal curves.

Our results for nonordinary singularities are much less complete, if for no other reason than our lack of a reasonable classification. We mention just one:

CURVE 4 (CUSP). Let $A = k[x,y]/(x^3 = y^2) = k[t^2,t^3]$, and $I = (x,y)A$. Then

$$HC_n(A) = HC_n(k) \oplus V_{n+1} \oplus V_{n+1} \oplus (I\otimes\Omega_k^n)$$

and $K_n(A)$ maps onto $K_n(k)\oplus V_n\oplus V_n\oplus\Omega_k^n$ for $n\le 3$. Moreover, if the Conjecture mentioned above is correct, then for all n,

$$K_n(A) = K_n(k)\oplus V_n\oplus V_n\oplus\Omega_k^n.$$

Note that the groups $K_n(A)$ for $n = 0,1$ are known to agree with the above formula [**Kr**, 12.1].

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