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A WEAK RAMANUJAN CONJECTURE FOR GENERIC CUSPIDAL SPECTRUM OF QUASI-SPLIT GROUPS

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The strong form of the Ramanujan conjecture for a quasi-split reductive group had predicted that all the components of a cusp form are tempered (cf. [**5**, **13**]). But, examples of Kurokawa (cf. [**11**]) and Howe and Piatetski-Shapiro [**5**] have shown that this is not true in general. In fact, even for PSP_4 there are cusp forms which already defy the conjecture. Consequently, in [**11**] Langlands predicted that an automorphic representation fails to be tempered only if it lifts to an anomalous representation of some $\mathrm{GL}(n)$.

On a quasi-split group, one may consider the class of cusp forms which as representations can be realized on spaces of functions which transform on the left according to a generic character of the unipotent radical of a Borel subgroup (i.e., the ones with Whittaker models). We call such automorphic representations *generic*. None of the nontempered automorphic representations constructed so far are generic. In what follows, we shall produce some evidence towards the validity of the strong Ramanujan conjecture for generic automorphic forms (Theorems 1 and 2, and Corollary 2). In fact, in Corollary 2, we obtain a uniform bound for the Hecke eigenvalues of generic cusp forms on many absolutely simple quasi-split groups over number fields. It also provides us with a new proof of the best available estimate for the Fourier coefficients of Maass wave forms (Corollary 4). Detailed proofs will appear elsewhere.

Let $\mathbf{G}$ be a quasi-split group over a number field F. Set $G = \mathbf{G}(\mathbf{A}_F)$. Let $\mathbf{P}$ be a maximal F-parabolic subgroup of $\mathbf{G}$. Write $\mathbf{P} = \mathbf{MN}$ and let P, M, and N be the corresponding groups of adelic points. Let v be a place of F and denote by G_v, P_v, M_v, and N_v the corresponding local groups of F_v-rational points. Let σ be a cusp form on M and write $\sigma = \bigotimes_v \sigma_v$, a restricted tensor product of representations of the local groups G_v (cf. [**3**]).

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Let $\mathbf{U}$ be the unipotent radical of a Borel subgroup of $\mathbf{G}$ such that $\mathbf{U} \supset \mathbf{N}$. A character $\chi = \bigotimes_v \chi_v$ of $\mathbf{U}(\mathbf{A}_F)$ is called *generic* if, for every v, the restriction of χ_v to every root space in U_v generated by a simple root is nontrivial. The representation σ_v is then called χ_v-generic, if it can be realized on a space of functions W on G_v satisfying

$$W(ug) = \chi_v(u)W(g) \qquad (u \in U_v,\ g \in G_v).$$

We shall say σ is χ-generic if each σ_v is χ_v-generic (cf. [**15**], for example). Finally, we say σ is generic, if it is generic with respect to some χ.

Let α be the unique reduced F-root of the split component $\mathbf{A}$ of the center of $\mathbf{M}$ in $\mathbf{N}$ and denote by ρ_P half the sum of the F-roots generating $\mathbf{N}$. Then $\tilde{\alpha} = \langle \rho_P, \alpha \rangle^{-1} \rho_P$ belongs to the complex dual of the real Lie algebra $\mathfrak{a}$ of $\mathbf{A}$. Let $\mathbf{C}$ be the field of complex numbers. We shall now identify $\mathbf{C}$ with a subspace of the complex dual of $\mathfrak{a}$ by identifying $s \in \mathbf{C}$ with $s\tilde{\alpha}$. Then for each v, s becomes an element in the complex dual of the real Lie algebra $\mathfrak{a}_v$ of the split torus in the center of $\mathbf{M}$ as a group over F_v.

From [**16**], for every v there exists a homomorphism H_{P_v} from M_v into $\mathfrak{a}_v$. Denote by

$$I(s, \sigma_v) = \operatorname*{Ind}_{P_v \uparrow G_v} \sigma_v \otimes q_v^{\langle s, H_{P_v}(\ )\rangle},$$

the representation of G_v induced from σ_v and s, where q_v is the cardinality of the residue class field of F_v.

Let $A(s, \sigma_v)$ be the standard intertwining operator attached to $I(s, \sigma_v)$ (cf. [**15, 16**]). Moreover, given a cusp form σ on M, let $M(s, \sigma)$ be the constant term of the Eisenstein series defined by σ (cf. [**8**]). Then $M(s, \sigma) = \bigotimes_v A(s, \sigma_v)$.

We use S to denote a finite set of places of F, including the archimedean ones, such that for $v \notin S$, $\mathbf{G}, \sigma_v$, and χ_v are all unramified (cf. [**2**]).

Let ${}^L M$ be the L-group of $\mathbf{M}$ (cf. [**2, 10**]). It is a complex Lie group. Given a finite-dimensional complex representation r of ${}^L M$, let r_v be its restriction to ${}^L M_v$, the L-group of $\mathbf{M}$ as a group over F_v.

For a finite place v of F with $v \notin S$, let $L(s, r_v, \sigma_v)$ be the Langlands L-function attached to r_v and σ_v (cf. [**2, 10**]). Here s is a complex number. Then the Euler product

$$L_S(s, r, \sigma) = \prod_{v \notin S} L(s, r_v, \sigma_v)$$

always converges absolutely for $\mathrm{Re}(s)$ large enough [**10**].

There exist m finite-dimensional representations $r_1, \dots, r_m$ of ${}^L M$ such that (cf. [**9, 15**])

$$(1)\quad M(s,\sigma)f = \bigotimes_{v \in S} A(s, \sigma_v)\tilde{f}_v \otimes \bigotimes_{v \notin S} f_v \cdot \prod_{i=1}^{m} L_S(is, \tilde{r}_i, \sigma)/L_S(1 + is, \tilde{r}_i, \sigma),$$

where $f = \bigotimes_v f_v$, $f_v \in I(s, \sigma_v)$, and for every $v \notin S$, f_v is the $\mathbf{G}(0_v)$-fixed function, normalized by $f_v(e_v) = 1$. Here $\tilde{r}_i$ is the contragredient of r_i. The representations r_i are all irreducible. The significance of these L-functions is that all the automorphic L-functions studied so far are among them. We then prove:

THEOREM 1. *Suppose σ is generic and cuspidal.*

(a) *Assume $m = 1$. Then for* $\mathrm{Re}(s) > 2$, *the L-function $L_S(s, r_1, \sigma)$ is absolutely convergent.*

(b) *Suppose $m > 1$. Assume further that the restriction of σ to the center of M is trivial. Then for* $\mathrm{Re}(s) > 2$, *the L-functions $L_S(s, r_i, \sigma)$ are all absolutely convergent, $i = 1, \ldots, m$.*

This is a consequence of the following result.

THEOREM 2. *Suppose σ is cuspidal and generic.*

(a) *Assume $m = 1$. Then for each finite $v \in S$, $A(s, \sigma_v)$ is holomorphic for* $\mathrm{Re}(s) \geq 1$.

(b) *Suppose $m > 1$. Assume further that the restriction of σ to the center of M is trivial. Then for each finite $v \in S$, $A(s, \sigma_v)$ is holomorphic for* $\mathrm{Re}(s) \geq 1$.

REMARK 1. When $\mathbf{G} = \mathbf{GL}_{n+r}, \mathbf{M} = \mathbf{GL}_n \times \mathbf{GL}_r$, $m = 1$, and Theorem 2 becomes Proposition 2.2 of [**6**], whose proof is based on a classification theorem for generic representations of $\mathbf{GL}_n$ which is a fairly deep result of Bernstein and Zelevinski [**1**, **17**] (cf. [**7**]), and certain properties of local Rankin-Selberg L-functions [**7**].

REMARK 2. The assumption that σ is trivial on the center of M (when $m > 1$) is made so that the necessary induction can be established in general (Lemma 1). However there are many cases with $m > 1$ for which the assumption is not necessary. This is due to our better understanding of $L_S(s, r_i, \sigma)$, $i = 2, \ldots, m$, in these special cases (cf. Corollary 2).

REMARK 3. If σ is trivial on the center of M, then Theorem 1 is still true if $\mathbf{M}$ is replaced by its adjoint group $\overline{\mathbf{M}}, \rho\colon \mathbf{M} \to \overline{\mathbf{M}}$, since then σ may be extended to a representation $\bar{\sigma}$ of $\overline{\mathbf{M}}(\mathbf{A}_F)$ satisfying

$$L_S(s, r_i, \sigma) = L_S(s, r_i \cdot {}^L\rho, \bar{\sigma}),$$

$i = 1, \ldots, m$.

COROLLARY 1. *Suppose σ is cuspidal and generic.*

(a) *Assume $m = 1$. Then for* $\mathrm{Re}(s) \geq 1$, *the poles of the corresponding Eisenstein series depend only on the poles of $\bigotimes_{v=\infty} A(s, \sigma_v)$ and $L_S(s, \tilde{r}_1, \sigma)$.*

(b) *Suppose $m > 1$. Assume further that the restriction of σ to the center of M is trivial. Then for* $\mathrm{Re}(s) \geq 1$, *the poles of the corresponding Eisenstein series depend only on the poles of $\bigotimes_{v=\infty} A(s, \sigma_v)$ and $\prod_{i=1}^{2} L_S(is, \tilde{r}_i, \sigma)$. Moreover, for* $\mathrm{Re}(s) > 1$, *the dependence on the L-functions reduces to $L_S(s, \tilde{r}_1, \sigma)$.*

COROLLARY 2. *Let $\mathbf{G}$ be either $\mathbf{GL}(n)$, $\mathbf{U}(n, n)$, $\mathbf{U}(n+1, n)$, $\mathbf{SP}(2n)$, $\mathbf{SO}(m, n)$, $m = n$, $n+1$, $n+2$, their groups of similitudes, their adjoint groups and those of split groups of types E_6 and E_7 over a number field F. Let $\sigma = \bigotimes_v \sigma_v$ be a χ-generic cusp form on $G = \mathbf{G}(\mathbf{A}_F)$. At each place v of F where σ_v, χ_v, and $\mathbf{G}$ are all unramified, let t_v be the element in ${}^LT^0$ representing* *the corresponding semisimple conjugacy class in* ${}^LG_v$. *Finally, let* μ *be a weight for the restriction of the standard representation (first fundamental representation) of* ${}^LG$ *to* ${}^LG^0$. *Then* $|\mu(t_v)| < q_v$, *where* q_v *is the cardinality of the residue class field of* F_v.

REMARK 1. The strong Ramanujan Conjecture is equivalent to $|\mu(t_v)| \leq 1$.

REMARK 2. When $\mathbf{G} = \mathbf{GL}_n$, the proof of Theorem 2 (see Lemma 3 below) applied to $\mathbf{GL}_{2n}$ with Levi factor $\mathbf{M} = \mathbf{GL}_n \times \mathbf{GL}_n$ and representation $\sigma \otimes \sigma$ of M, will immediately imply that $|\mu(t_v)| < q_v^{1/2}$. This is Corollary 2.5 of [**7**].

SKETCH OF THE PROOF OF THEOREM 2. Under the assumption in part (b), we may consider the L-functions as those for the adjoint group $\overline{\mathbf{M}}$ of $\mathbf{M}$. Let $\rho\colon \mathbf{M} \to \overline{\mathbf{M}}$ be the corresponding projection.

In the following lemma we shall assume that $\mathbf{G}$ has neither a factor of type ${}^2A_{2k}$, nor a factor of type F_4, if the part of $\mathbf{M}$ in this factor is generated by $\{\alpha_1, \alpha_3, \alpha_4\}$ (α_1 and α_2 are long). In these two cases the necessary lemmas can be verified directly.

LEMMA 1. *Fix* i, $2 \leq i \leq m$. *There exist a quasi-split connected reductive* F*-group* $\mathbf{G}_i$, *unramified outside* S, *a maximal* F*-parabolic subgroup with a Levi factor* $\mathbf{M}_i \subset \mathbf{G}_i$ *whose adjoint group* $\overline{\mathbf{M}}_i$ *embeds into* $\overline{\mathbf{M}}$ *by a* F*-rational map* $j\colon \overline{\mathbf{M}}_i \to \overline{\mathbf{M}}$ *such that if* $r'_1, \ldots, r'_{m_i}$ *are the corresponding representations of* ${}^LM_i$, *then*

$$r_i \cdot {}^L\rho = r'_1 \cdot {}^L\rho_i \cdot {}^Lj,$$

where $\rho_i\colon \mathbf{M}_i \to \overline{\mathbf{M}}_i$ *is the natural projection. Moreover* $m_i < m$.

COROLLARY 3 (OF LEMMA 1). *Let* $\overline{\mathbf{M}}$ *be the adjoint group of* $\mathbf{M}$. *Fix a generic cusp form on* $\overline{\mathbf{M}}(\mathbf{A}_F)$. *Then every* L*-function* $L_S(s, r_i, \sigma)$, $i = 1, \ldots, m$, *extends to a meromorphic function of* s *on* $\mathbf{C}$ *satisfying a standard functional equation.*

REMARK. It is the subject of a forthcoming paper that for a large class of these partial L-functions, local factors at the ramified places can be defined in such a way that each resulting L-function extends to a meromorphic function of s with possibly only a *finite number* of poles in the entire complex plane. The factors at the archimedean places are the Artin factors defined by the Langlands' local class field theory at such places.

Up to a finite number of factors (which can be made nonzero), every nonconstant Fourier coefficient of the Eisenstein series (cf. [**14, 15**]) is equal to

$$\prod_{i=1}^{m} L_S(1+is, \tilde{r}_i, \sigma)^{-1},$$

and consequently, for $\mathrm{Re}(s) \geq 0$,

$$\prod_{i=1}^{m} L_S(1+is, \tilde{r}_i, \sigma)$$

has only a finite number of zeros which are all on the real axis. Now, by Lemma 1 and induction we in fact have:

LEMMA 2. *For* $\mathrm{Re}(s) \geq 1$, $L_S(s, \tilde{r}_i, \sigma)$ *and the quotient*

$$L_S(s, \tilde{r}_i, \sigma)/L_S(1 + s, \tilde{r}_i, \sigma)$$

both have only a finite number of poles and zeros, $1 \leq i \leq m$.

PROOF OF THEOREM 2. By Lemma 2 and relation (1), we conclude that except for a finite number of real poles, $M(s, \sigma)f$ and $\bigotimes_{v \in S} A(s, \sigma_v)f_v$ have the same poles when $\mathrm{Re}(s) \geq 1$. Fix $v \in S$, $v < \infty$. For every $w \in S$, $w \neq v$, $w < \infty$, we can choose f_w such that $A(s, \sigma_w)f_w$ becomes a nonzero constant independent of s. Now, suppose for some s, with $\mathrm{Re}(s) \geq 1$, $A(s, \sigma_v)$ has a pole. For each such pole and each $w = \infty$, choose f_w such that $A(s, \sigma_w)f_w \neq 0$. The operator $A(s, \sigma_v)$, being a rational function of q_v^{-s}, will then have infinitely many poles parallel to the imaginary axis. Consequently, $M(s, \sigma)f$ must also have infinitely many such poles. This is a contradiction to the finiteness of poles for $M(s, \sigma)$ when $\mathrm{Re}(s) \geq 0$.

To prove Theorem 1, we need the following lemma.

LEMMA 3. *Suppose v is unramified. Then for* $\mathrm{Re}(s) \geq 1$, *each* $L(s, \tilde{r}_{i,v}, \sigma_v)$ *is holomorphic,* $1 \leq i \leq m$.

To conclude we shall now give a new proof of the following result (cf. [**12**] for the original proof; it is also due to Serre). It no longer requires the use of Rankin products; nor of Landau's Lemma.

COROLLARY 4. *Let π be a nonmonomial cuspidal representation of* $\mathbf{PGL}_2(\mathbf{A}_F)$. *At each unramified v, let* $t_v = \mathrm{diag}(\alpha_v, \alpha_v^{-1})$ *be the corresponding semisimple conjugacy class in* $\mathbf{SL}_2(\mathbf{C})$. *Then* $q_v^{-1/5} < |\alpha_v| < q_v^{1/5}$.

PROOF. We only need to apply Lemma 3 to a split group of type F_4 (example 4.1.6 of [**15**]) with $\mathbf{M}$ generated by $\{\alpha_1, \alpha_2, \alpha_4\}$ (α_1 and α_2 are long), and $\sigma = \Pi \otimes \pi$, where Π is the Gelbart-Jacquet [**4**] lift of π to $\mathbf{PGL}_2(\mathbf{A}_F)$.

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