BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 15, Number 1, July 1986

PERIODIC GEODESICS OF GENERIC NONCONVEX DOMAINS IN $\mathbf{R}^2$ AND THE POISSON RELATION

VESSELIN M. PETKOV AND LUCHEZAR STOJANOV

1. Introduction. Let $\Omega \subset \mathbf{R}^n$, $n \geq 2$, be a bounded connected domain with C^∞ smooth boundary $\partial\Omega$. Consider the eigenvalues $\{\lambda_j^2\}_{j=1}^\infty$ corresponding to the Dirichlet problem for the Laplacian

$$(1) \qquad -\Delta u = \lambda^2 u \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

The Poisson relation for $\sigma(t) = \sum_j \cos \lambda_j t$ has the form

$$(2) \qquad \operatorname{singsupp} \sigma(t) \subset \bigcup_{\gamma \in \mathcal{L}_\Omega} \{-T_\gamma\} \cup \{0\} \cup \bigcup_{\gamma \in \mathcal{L}_\Omega} \{T_\gamma\}.$$

Here $\mathcal{L}_\Omega$ is the union of all generalized periodic geodesics γ in $\overline{\Omega}$, including those lying entirely on $\partial\Omega$, and T_γ is the period (length) of γ (see [**1**]). Generalized geodesics are projections on $\overline{\Omega}$ of the generalized bicharacteristics of $\partial_t^2 - \Delta$, introduced by Melrose and Sjöstrand [**6**]. We have proved in [**8, 9**] that for generic strictly convex domains in $\mathbf{R}^2$ the relation (2) becomes an equality and the spectrum of (1) determines the lengths of all periodic geodesics (see [**5**] for related results). The purpose of this announcement is to prove the same result for generic nonconvex domains in $\mathbf{R}^2$.

2. Main results. In the analysis of (2) for nonconvex domains three difficulties appear: (A) the existence of periodic geodesics having gliding segments on $\partial\Omega$ and linear segments in the interior of Ω, (B) some linear segment l of a periodic geodesic could be tangent to $\partial\Omega$ at some interior point of l, (C) the linear Poincaré map P_γ of a reflecting periodic geodesic γ could contain in its spectrum 1 or $\sqrt[p]{1}$ with $p \in \mathbf{N}$. We refer to [**3**] for the precise definition of reflecting geodesics and the related Poincaré map. A linear segment is a set $l = [x, y] = \{z; z = \alpha x + (1-\alpha)y,\ 0 \leq \alpha \leq 1\}$, while a gliding segment is an arc $\delta \subset \partial\Omega$. We show below that generically for domains in $\mathbf{R}^2$ the phenomena (A), (B), (C) cannot occur. We begin by assuming $\Omega \subset \mathbf{R}^2$.

Set $\partial\Omega = X$ and consider the space $C^\infty_{\text{emb}}(X, \mathbf{R}^2)$ of all C^∞ smooth embeddings of X into $\mathbf{R}^2$ with the Whitney topology [**2**]. For $f \in C^\infty_{\text{emb}}(X, \mathbf{R}^2)$ we denote by $\Omega_f \subset \mathbf{R}^2$ the bounded domain with boundary $f(X)$. A set $\mathcal{R} \subset C^\infty_{\text{emb}}(X, \mathbf{R}^2)$ will be called residual if $\mathcal{R}$ is a countable intersection of open dense sets.

THEOREM 1. *Let Ω be a domain with boundary X. There exists a residual set $\mathcal{R} \subset C^\infty_{\text{emb}}(X, \mathbf{R}^2)$ such that for every $f \in \mathcal{R}$ there are no generalized periodic geodesics $\gamma \in \mathcal{L}_{\Omega_f}$ having at least one gliding segment on $f(X)$ and*

Received by the editors August 26, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 58G25, 35R30.

at least one linear segment in the interior of Ω_f. *Moreover, for* $f \in \mathcal{R}$ *every reflecting geodesic* $\gamma \in \mathcal{L}_{\Omega_f}$ *has Poincaré map* P_γ *whose spectrum does not contain* $\sqrt[p]{1}$ *for every* $p \in \mathbf{N}$.

REMARK 1. The above result has been conjectured in [**9**]. For generic strictly convex domains in $\mathbf{R}^2$ the conclusion concerning Poincaré map was established by Lazutkin [**4**].

THEOREM 2. *Let* Ω *be a domain with boundary* X. *There exists a residual set* $\mathcal{R} \subset C^\infty_{\text{emb}}(X, (\mathbf{R}^2)$ *such that for every* $f \in \mathcal{R}$ *there are no generalized periodic geodesics* $\gamma \in \mathcal{L}_{\Omega_f}$ *containing at least one linear segment* l *tangent to* $f(X)$ *at some interior point of* l.

REMARK 2. According to Theorems 1 and 2, for generic domains in $\mathbf{R}^2$ every periodic geodesic, different from the boundary, is a reflecting one. The above assertion about Poincaré map and Theorem 2 admit a generalization for domains in $\mathbf{R}^n$ which will be published elsewhere.

Combining the rational independence of periods of reflecting geodesics for generic domains, established in [**8, 9**], Theorems 1 and 2 and the result in [**3**], we obtain

THEOREM 3. *Under the assumptions and notations of Theorem* 1, *for every* $f \in \mathcal{R}$ *the Poisson relation* (2) *becomes an equality where* $\sigma(t)$ *is related to the eigenvalues for problem* (1) *in* Ω_f *with boundary condition on* $f(X)$ *and the unions in* (2) *are taken over all generalized periodic geodesics in* $\mathcal{L}_{\Omega_f}$.

3. Idea of the proof of Theorem 1. Let $f \in C_{\text{emb}}(X, \mathbf{R}^2)$ and let γ be a generalized geodesic in $\mathcal{L}_{\Omega_f}$ having linear segments in $\mathbf{R}^2 \backslash f(X)$. Assume γ antisymmetric, that is γ does not contain a linear segment l orthogonal to $f(X)$ at some end point of l. In this case there are different points $y_i = f(x_i)$, $i = 1, \dots, s$ on $f(X)$, an integer $k \geq s$ and a surjection $\omega\colon \{1, \dots, k\} \to \{1, \dots, s\}$ with $\omega(1) = 1$, $\omega(2) = 2$, $\omega(k) = s$, so that the linear segments $l_j = [y_{\omega(j)}, y_{\omega(j+1)}]$, $j = 1, \dots, k-1$ are successive segments of γ with reflection points $y_{\omega(j)}$, $j = 2, \dots, k-1$, the curvatures of $f(X)$ at y_1 and y_k vanish and l_1 and l_{k-1} are tangent to $f(X)$ at y_1 and y_k respectively. Setting $\omega(1) = \omega(k+1)$, we have $\omega(i) \neq \omega(i+1)$ for $i = 1, \dots, k$ and $\{\omega(i), \omega(i+1)\} \neq \{\omega(j), \omega(j+1)\}$ whenever $1 \leq i < j \leq k-1$. The maps having the properties listed above will be called admissible antisymmetric. Let $Z^{(s)} = \{(z_1, \dots, z_s) \in Z^s; z_i \neq z_j \text{ for } i \neq j\}$. For $i = 1, \dots, s$, set $I_i = \{j;$ there exists $t = 1, \dots, k-1$ with $\{i, j\} = \{\omega(t), \omega(t+1)\}\}$ and denote by U_ω the set of those $z \in (\mathbf{R}^2)^{(s)}$ such that $z_i \notin$ convex hull $\{z_j; j \in I_i\}$ for every $i = 1, \dots, s$. Finally, consider the map $F\colon U_\omega \to \mathbf{R}$ given by

$$F(z) = \sum_{i=1}^{k-1} \|z_{\omega(i)} - z_{\omega(i+1)}\|.$$

It is clear that $x' = (x_2, \dots, x_{s-1})$ is a critical point of $F \circ f^s(x_1, z', x_s)$ considered as a function of $z' = (z_2, \dots, z_{s-1}) \in X^{(s-1)}$, where $f^s(x) = (f(x_1), \dots, f(x_s))$. Fix k, s, F and an admissible antisymmetric map ω and denote by T_ω the set of those $f \in C^\infty_{\text{emb}}(X, \mathbf{R}^2)$ such that if $x = (x_1, \dots, x_s) \in$

$X^{(s)}$, $f^s(x) \in U_\omega$, $\operatorname{grad}_{x'}(F \circ f^s)(x) = 0$ and the curvatures of $f(X)$ at $f(x_1)$ and $f(x_s)$ vanish, then we have $\langle f(x_2) - f(x_1), n_{x_1} \rangle = 0, n_{x_1}$ being the normal to $f(X)$ at x_1 and $\langle \, , \, \rangle$ the scalar product in $\mathbf{R}^3$. Our aim is to show that T_ω is residual in $C^\infty_{\text{emb}}(X, \mathbf{R}^2)$. To do this, we use the s-fold bundle of the 2-jets. Namely, let $\alpha\colon J^2(X, \mathbf{R}^2) \to \mathbf{R}^2$ and $\beta\colon J^2(X, \mathbf{R}^2) \to \mathbf{R}^2$ be the source and the target maps (see [**2**]). Set

$$M = (\alpha^s)^{-1}(X^{(s)}) \cap (\beta^s)^{-1}(U_\omega) \cap V,$$

where V is the set of those $(j^2 f_1(x_1), \ldots, j^2 f_s(x_s)) \in (J^2(X, \mathbf{R}^2))^s$ with $df_i(x_i) \neq 0$ for every $i = 1, \ldots, s$. Clearly, M is an open submanifold of $J^2_s(X, \mathbf{R}^2) = (\alpha^s)^{-1}(X^{(s)})$. To describe the above situation, we introduce the set Σ of those $\sigma = (j^2 f_1(x_1), \ldots, j^2 f_s(x_s)) \in M$ such that $\operatorname{grad}_{x'}(F \circ f^s)(x) = 0$, the curvature of $f_1(X)$ at $f_1(x_1)$ and that of $f_s(X)$ at $f_s(x_s)$ vanish and the vector $f_2(x_2) - f_1(x_1)$ is collinear with the tangent to $f_1(X)$ at $f_1(x_1)$. The main difficulty is to show that Σ is a smooth submanifold of M with $\operatorname{codim} \Sigma = s + 1$. Therefore, by applying the multijet transversality theorem in [**2**], we prove that T_ω is residual in $C^\infty_{\text{emb}}(X, \mathbf{R}^2)$. Similarly we treat admissible symmetric maps ω which are related to geodesics on $f(X)$ having segments l orthogonal to $f(X)$ at some end point $y \in f(X)$ of l. Then $\bigcap_\omega T_\omega$, where ω runs over all admissible maps, is residual in $C^\infty_{\text{emb}}(X, \mathbf{R}^2)$.

For the proof of the second part of Theorem 1 we use essentially the representation of Poincaré map P_γ related to a reflecting geodesic γ, found by Petkov and Vogel [**7**]. We introduce a corresponding singular set Σ_1 and again the main point is to prove that Σ_1 can be covered by a countable union of smooth manifolds having codimension $s + 1$.

A similar approach is used for the proof of Theorem 2.

REFERENCES

1. K. Andersson and R. Melrose, *The propagation of singularities along gliding rays*, Invent. Math. **41** (1977), 197–232.

2. M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Graduate Texts in Math., vol. 14, Springer-Verlag, Berlin and New York, 1973.

3. V. Guillemin and R. Melrose, *The Poisson summation formula for manifolds with boundary*, Adv. in Math. **32** (1979), 204–232.

4. V. F. Lazutkin, *Convex billiard and eigenfunctions of the Laplace operator*, Leningrad University, 1981. (Russian)

5. S. Marvizi and R. Melrose, *Spectral invariants of convex planar regions*, J. Differential Geom. **17** (1982), 475–502.

6. R. Melrose and J. Sjöstrand, *Singularities in boundary value problems*. I, II, Comm. Pure Appl. Math. **31** (1978), 593–617; **35** (1982), 129–168.

7. V. Petkov and P. Vogel, *La représentation de l'application de Poincaré correspondant aux rayons périodiques réflechissants*, C. R. Acad. Sci. Paris, Sér. A **296** (1983), 633–635.

8. V. Petkov, *Propriétés génériques des rayons réflechissants et applicationns aux problèmes spectraux*, Séminaire Bony-Sjöstrand-Meyer, École Polytechnique, Centre de Mathématiques, Exposé XII, 1984–1985.

9. V. Petkov and L. Stojanov, *Periods of multiple reflecting geodesics and inverse spectral results*, preprint.

INSTITUTE OF MATHEMATICS, BULGARIAN ACADEMY OF SCIENCES, P. O. BOX 373, 1090 SOFIA, BULGARIA