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ON THE ZEROS OF MAASS WAVE FORM L-FUNCTIONS

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In [**1**], Epstein, Sarnak, and I gave some general results concerning the zeros of an L-function attached to an even Maass wave form for $\Gamma = \mathrm{PSL}(2, \mathbf{Z})$. The main result was that such an L-function has many zeros on its critical line. In fact, we showed that the analogue of the Hardy-Littlewood theorem for the Riemann zeta function holds for these L-functions as well. In this note I announce the next step: the analogue of Selberg's theorem. That is, a positive proportion of the zeros lie on the critical line. This then significantly extends the class of Dirichlet series for which this is known. To date, this included only the classical Dirichlet L-series [**8**] including the Riemann zeta function [**7**], and the L-functions attached to holomorphic cusp forms of Γ [**2, 4, 5**]. (The proofs in the last case and the present case probably extend to cusp forms on congruence subgroups as well, but this has not been thoroughly verified.) Effectively then, this result applies to all cusp forms for $\mathrm{GL}(2)$.

Let us formulate the theorem more explicitly. We begin with Γ acting on the upper half-plane $\mathcal{H} = \{z \in \mathbf{C} \colon \operatorname{Im} z > 0\}$ via linear fractional transformations. A Maass wave form is a Γ-automorphic function on $\mathcal{H}$ which is in $L^2(\Gamma\backslash\mathcal{H})$ and is simultaneously an eigenfunction of the Laplacian and all the Hecke operators. That is, f satisfies

(i) $$\int_{\Gamma\backslash\mathcal{H}} |f(z)|^2 \frac{dxdy}{y^2} < \infty,$$

(ii) $$\Delta f = (\frac{1}{4} + r^2) f, \quad \Delta = -y^2(\partial_x^2 + \partial_y^2),$$

(iii) $$f(\gamma z) = f(z), \quad \gamma \in \Gamma, \quad z \in \mathcal{H},$$

(iv) $$T_n f = a(n) f, \quad n \geq 1.$$

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Here T_n is the nth Hecke operator defined (for weight zero) by

$$T_n f(z) = n^{-1/2} \sum_{ad=n} \sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right).$$

We shall further assume that f is even, i.e.

(v) $$T_{-1} f(z) = f(-\bar{z}) = f(z).$$

The case of odd f ($T_{-1}f = -f$) can be handled in a similar manner.

There are two standard normalizations for f; either the integral in (i) is one or the Fourier coefficients of f are exactly the $a(n)$. We assume the latter so that

$$f(z) = \sum_{n=1}^{\infty} a(n) y^{1/2} K_{ir}(2\pi n y) \cos(2\pi n x).$$

Here K_{ir} is the modified Bessel function of the third kind and r comes from the eigenvalue in (ii).

We are concerned with the zeros of the function defined in $\sigma = \operatorname{Re} s > 1$ by either

$$L_f(s) = \sum_{n=1}^{\infty} a(n) n^{-s} = \prod_p \left(1 - a(p)p^{-s} + p^{-2s}\right)^{-1}.$$

As noted in [**1**], $L_f(s)$ is entire and satisfies the functional equation

$$\xi_f(s) = (2\pi)^{-s} \Gamma_r(s) L_f(s) = \xi_f(1-s),$$

where

$$\Gamma_r(s) = \int_0^{\infty} K_{ir}(y) y^{s-1}\, dy$$
$$= 2^{s-2} \Gamma\left(\frac{s+ir}{2}\right) \Gamma\left(\frac{s-ir}{2}\right).$$

In the usual way, one deduces that $L_f(s)$ has no zeros in $\sigma \geq 1$, has trivial zeros at $s = -2n \pm ir$, $n = 0, 1, 2, \ldots$, in $\sigma \leq 0$ and has all other zeros in the strip $0 < \sigma < 1$. The Riemann hypothesis here is, of course, that all the zeros lie on the line $\sigma = 1/2$.

Letting

$$N(T) = \#\{\rho = \beta + i\gamma \colon\ 0 < \gamma \leq T,\ 0 < \beta < 1,\ L_f(\rho) = 0\}$$

and

$$N_0(T) = \#\{\rho = \tfrac{1}{2} + i\gamma \colon\ 0 < \gamma \leq T,\ L_f(\rho) = 0\},$$

we have

$$N(T) \sim \frac{2}{\pi} T \log T,$$

and from [**1**] for some positive constant A

$$N_0(T) > AT.$$

Here we announce the following.

THEOREM. *For $L_f(s)$ as above, there is a positive constant A such that*

$$N_0(T) > AT \log T.$$

REMARK. The A depends on f in some subtle but calculable way. In particular it depends on the eigenvalue $\frac{1}{4} + r^2$. However, no attempt is made in the proof to give it a numerical value, or to show this dependence.

The main outline of the proof is essentially the Selberg-Titchmarsh method as modified in [**4**, **5**] for holomorphic forms. There are again some new difficulties.

The first is that because the form f has weight zero its Mellin transform $\xi_f(s)$ decays too rapidly on the critical line to allow for a sensitive detection of zeros. This is corrected in a seemingly ad-hoc manner by introducing a growth factor. One studies

$$\xi_f^*(s) = \xi_f(s)\left(s - \tfrac{1}{2}\right)$$

which decays slightly less rapidly, but still has functional equation

$$\xi_f^*(s) = -\xi_f^*(1-s),$$

and critical line $\sigma = 1/2$. This function actually arises naturally as the Mellin transform of the weight two form $2i(\partial f/\partial z)$.

The second main difficulty is essentially the same as in the holomorphic case, though the methods for overcoming it are quite different. We need to estimate sums of the type

$$\sum_{N \geq 1} \sum_{n} a(n) a\left(\frac{cn+N}{d}\right), \qquad (c,d) = 1. \tag{1}$$

For the holomorphic case this was handled by analyzing the Dirichlet series

$$\sum_{n=1}^{\infty} a(n) a\left(\frac{cn+N}{d}\right) n^{-s}$$

and obtaining explicit estimates in N, c, d, $|s|$ in regions of nonabsolute convergence (to the left of $\sigma = 1$). See [**3**]. This relied heavily on two things. First, it required explicit computations involving the spectrum of the Laplacian for congruence subgroups (to deal with the c-d dependence). Second, it uses the fact that the holomorphic forms of fixed weight can be represented as a sum of Poincaré series.

Unfortunately, the latter technique is not available to us here. This difficulty is overcome by some methods of Kuznetsov [**6**]. In that paper he deals with sums of the type in (1) with $c = d = 1$. Essentially, he replaces one of the coefficients $a(n)$ by a sum of Kloosterman sums, uses some complicated inversion formulas and "reduces" the sum in question to multifold sums of multiple integrals, with sums over the full spectrum of the Laplacian. Then careful estimates are required to show convergence in all the relevant parameters. In our case, we need to do essentially the same thing, but over congruence groups as well. The computations are technically complicated and quite long.

To study this problem in the most natural way, i.e., via (2), would require an estimate of the type

$$\int_{\Gamma\backslash\mathcal{H}} |f(z)|^2 u_\kappa(z)\frac{dxdy}{y^2} \ll e^{-\pi\kappa/2}\kappa^A\|u_\kappa\|_2,$$

where u_κ is an arbitrary Maass wave form with eigenvalue $\frac{1}{4}+\kappa^2$, and A is some positive constant. At this time, this seems unattainable.

The details of the proof of the theorem will appear elsewhere.

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