moments of distributions, finding Gaussian quadrature rules, expressing the solutions of differential equations as power series or as series of orthogonal polynomials, and evaluating and transforming such series.

The last part of the book is concerned with nonlinear multidimensional recurrences and iterations. In contrast with the earlier section, where the emphasis was on the generation of sequences and on the questions of stability and convergence of the backward recurrence as the starting point increased, the principal interest with these recurrences is the behavior of the sequences which are generated as a function of the initial values. Although the strange behavior of sequences which do not converge has attracted considerable recent interest, the field is too new for definitive treatment, and cases in which the sequences do converge to a limit are treated more fully. These include the classical Gauss arithmetic-geometric mean algorithm for the complete elliptic integral, as well as the Borchardt and Bartky algorithms. These are of particular interest both because of the classic nature of the problems which they solve, including the rectification of the lemniscate and ellipse, but also because they provide approaches to evaluating general elliptic functions and integrals which are not hypergeometric and do not satisfy linear differential equations.

In summary, the numerical mathematician concerned with evaluation of special functions will find most of this book of exceptional value, while the mathematician interested in other topics will be introduced to many surprising results, which draw on a wide spectrum of classical mathematical techniques.

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Modern dimension theory (second edition, revised and extended), by J. Nagata, Sigma Series in Pure Mathematics, vol. 2, Heldermann-Verlag, Berlin, 1983, 68.00 DM, x + 284 pp. ISBN 3-88538-002-1

Dimension theory is one of the triumphs of point-set topology. When Cantor showed that Euclidean spaces of different dimensions nevertheless admitted one-one correspondences, and Peano showed that this could even happen in a continuous way, the naive ideas about dimension were shattered. Was there even a topological invariant that could be called dimension? Brouwer showed that this was so, at least for Euclidean spaces; but his work did not lead to a satisfactory general theory. The key idea was contained in a remark of

Poincaré: Euclidean space R^3 is 3-dimensional because prison walls are 2-dimensional. This was developed, in the early 1920s, by Urysohn and independently by Menger, into a satisfactory theory of dimension. They defined what is now called the "small inductive dimension", ind X, of a topological space X, by: ind $\phi = -1$, and ind $X \leqslant n$ precisely when each point of X has arbitrarily small neighborhoods whose boundaries have ind $\leqslant n - 1$. At least for separable metrizable spaces this leads to a rich and rewarding theory. The following brief remarks can do no more than indicate the flavor of the subject, and a little of its history.

Among the more significant early results for separable metrizable spaces (emphasized, for example, in Menger's book [4]) one should mention:

THE SUBSPACE THEOREM. *If Y is a subspace of X, its dimension is at most that of X.*

THE SUM THEOREM. *If X is the union of a countable family of closed subspaces each of dimension at most n, then the dimension of X is at most n.*

THE SPLITTING THEOREM. *X is of dimension at most n if, and only if, it is the union of $n + 1$ (or fewer) subspaces of dimension* 0.

THE EMBEDDING THEOREMS. *If X is of dimension at most n, it is embeddable as a subspace in a compact metrizable space of the same dimension, and this is embeddable in Euclidean space R^{2n+1} (and in fact in a "universal n-dimensional" subspace of R^{2n+1}).*

The number $2n + 1$ is best possible (though this was not proved until somewhat later).

THE PRODUCT THEOREM. *If X is m-dimensional and Y is n-dimensional, then $X \times Y$ has dimension at most $m + n$.*

(The inequality here is counter-intuitive but inevitable; as P. Erdös showed, the rational points in Hilbert space form a 1-dimensional space whose product with itself is also 1-dimensional.)

And—most significant of all—*the dimension of R^n is n.*

These early results display a remarkable feature that has persisted in most later work: there is no restriction to compact or "smooth" spaces. Arbitrary (even pathological) separable metrizable spaces can be handled, and in fact turned to advantage—for instance, in exploiting the splitting theorem to reduce proofs of general theorems to the zero-dimensional case.

Other ways of defining dimension were also developed. Lebesgue introduced (in effect) what is now called the "covering dimension", dim, defined by: dim $X \leqslant n$ provided every finite open cover $\mathscr{U}$ of X has a refinement $\mathscr{V}$ for which no point belongs to more than $n + 1$ sets in $\mathscr{V}$. This says, roughly, that X can be approximated by polytopes (the nerves of suitable covers) of dimension at most n, and suggests that—for compact metrizable spaces at least—the dimension of X can be characterized by methods of algebraic topology. This was achieved by P. Alexandrov, who succeeded in characterizing dim X (for compact metrizable X) in terms of homology groups. (Nowadays one uses a cohomology version instead; it is neater and applies to more general spaces.) This algebraic approach led Pontrjagin to settle an old question by showing that inequality in the product theorem can occur even for compact spaces; in fact, as he showed, there is a 2-dimensional compactum whose product with itself has dimension 3.

Many other aspects of dimension have been investigated, for instance: continuous maps that raise (or lower) dimension; maps into spheres; infinite dimensionality of various kinds; other definitions of dimension that depend on the metric on the space; and a remarkable connection with measure theory, via Hausdorff dimension. An admirable account of the "classical" theory for finite-dimensional separable metrizable spaces (but omitting metric-dependent dimension functions) is provided by the book of Hurewicz and Wallman [**2**].

But inevitably one moves on to consider more general spaces; and there a major difficulty arises. As was pointed out in [**2**], the success of the "classical" theory is largely due to the fact that many different definitions of dimension coincide for separable metrizable spaces X; in particular, ind X = dim X = Ind X, the "large" inductive dimension (defined like ind but with "point" replaced by "closed subset"). For general spaces this is no longer the case. However, it was shown (independently) by Katětov and Morita in the 1950s that, at least for *metrizable* spaces, dim = Ind and a large part of the theory can be carried over to the nonseparable case—very roughly, everything not involving the originally defined "ind". That "ind" is inappropriate here was shown by a complicated example, due to Prabir Roy [**8**], of a complete metric space with dim = 1 and ind = 0. Large parts of the theory have been extended even further, mostly to spaces satisfying various strengthened forms of normality; some restrictions have to be made, since in the absence of normality the subspace and sum theorems need not hold. The subject has continued to be under active and rapid development, so a revised edition of Nagata's *Modern dimension theory* [**6**] is most opportune.

Like the first edition, this one focuses on metrizable spaces, in the main, with Ind (= dim) as the basic dimension function. After a brief introductory chapter, summarizing the necessary background from general topology, the book plunges into the detailed study of the fundamental properties of dimension in general metrizable spaces. The going is tough but rewarding. There follow chapters on mappings and dimension, and on the dimension of separable metrizable spaces, including some special properties of Euclidean spaces and the connection with measure. In the next chapter, dimension is characterized by the existence of special coverings and special metrics—for instance, the well-known theorem (of Ostrand and Nagata) that dim $X \leqslant n$ if and only if X has a compatible metric ρ such that, for every $n + 3$ points x, y_1, $y_2, \ldots, y_{n+2}$ of X there are distinct indices i, j such that $\rho(y_i, y_j) \leqslant \rho(x, y_i)$.

Chapter VI deals with infinite-dimensional spaces: large and small transfinite dimension (extending Ind and ind transfinitely), weak and strong infinite-dimensionality, and countable dimensionality are considered. Chapter

VII deals with nonmetrizable spaces. The sum theorem for dim is extended (following Ostrand) to locally finite systems of closed subsets in arbitrary spaces; then the chapter concentrates on normal (T_4) spaces. Here $\dim X \leqslant \operatorname{Ind} X$, and a significant part of the theory still holds for dim. The sum and subspace theorems for Ind, due to Dowker, come next; they assume "total normality" (that is, normality plus: each open set U has a cover by open F_σ sets that is locally finite in U). The final chapter, on dimension and cohomology, gives a rapid outline of the background in algebraic topology that is required, and proves (inter alia) that, for a finite-dimensional paracompact Hausdorff space X, $\dim X \leqslant n$ if and only if, for every integer $m \geqslant n$ and closed subset C of X, the natural homomorphism of $H^m(X)$ into $H^m(C)$ is surjective. The chapter concludes with a short discussion of Menger's old problem, now solved partly (but not completely), of finding a "good" (nonrecursive) axiomatic characterization of dimension.

It may be of interest to compare this book with some other standard treatments of dimension theory. The first book on the subject, by Menger [**4**], though written with pioneering enthusiasm, is perhaps now of mainly historical interest. The discussion in [**3**, §§25–28 and 45] covers the essentials of the "classical" separable metric case, in a tour-de-force of elegant compression. The treatment in [**2**] is probably still the best introduction to the subject for a neophyte. Nagami's book [**5**] is aimed mostly at normal, usually paracompact, spaces, though with excursions into metrizable spaces and, in particular, a useful treatment of metric-dependent dimension functions. It also includes an extensive treatment (by Y. Kodama) of cohomological dimension theory, with applications to the dimension of products and to generalized Cantor manifolds. (A nongeneralized Cantor manifold is a compactum of dimension n that cannot be disconnected by a subset of dimension $< n - 1$.) The book by Pears [**7**] also concentrates on general spaces, even more so than [**5**]; it includes a chapter on local dimension and explores a number of significant examples, including a careful exposition of Roy's example [**8**]. It ends with a detailed account of Katětov's functional-algebraic method for the dimension of metrizable spaces. Engelking's [**1**] is a textbook; it provides exercises and moves at a somewhat more leisurely pace, beginning with the separable metrizable case and going on to hereditarily normal and to compact spaces before ending with general metrizable spaces. Neither [**1**] nor [**7**] includes any algebraic topology.

The most meaningful comparison is of course with the first edition [**6**]. Besides many minor amplifications and clarifications of the arguments, and a number of simplifications, the new edition includes some interesting characterizations (due to Pontrjagin-Schnirelmann and to Janos) of dimension by global properties of covers; an improved treatment of dimension-characterizing metrics; and greatly extended treatments of infinite-dimensional spaces and of nonmetrizable spaces. The bibliography has been enlarged and modernized (to 1981). In fact, there is only one respect in which this second edition is not a distinct improvement on the first—namely, legibility. The present edition has been reproduced (and reduced) from typescript; this may have been unavoidable economically, but has the unfortunate consequence that the writing is too small and the symbols insufficiently varied. The first edition is much easier to read; but the present one is even more worth reading. It gives a very good account of its subject, and its title is well deserved.

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Gravitational curvature, an introduction to Einstein's theory, by Theodore Frankel, W. H. Freeman and Co., San Francisco, California, 1979, xviii + 172 pp., $8.95. ISBN 0-7167-1062-5

General relativity, an introduction to the theory of the gravitational field, by Hans Stephani, (edited by John Stewart; translated from German by Martin Pollock and John Stewart) Cambridge Univ. Press, New York, New York, 1982, xvi + 298 pp., $49.50. ISBN 0-521-24008-5

General relativity, by Robert M. Wald, University of Chicago Press, Chicago, Illinois, 1984, xiii + 491 pp., $50.00 HB; $30.00 PB. ISBN 0-266-87033-2

One hundred years ago there appeared in New York a book by William K. Clifford [7] containing the following passages:

(i) Our space is perhaps really possessed of a curvature varying from point to point, which we fail to appreciate because we are acquainted with only a small portion of space...

(ii) Our space may be really same (of equal curvature), but its degree of curvature may change as a whole with the time...

(iii) We may conceive our space to have everywhere a nearly uniform curvature, but that slight variations of the curvature may occur from point to point, and themselves vary with the time... We might even go so far as to assign to this variation of curvature of space 'what really happens in that phenomenon which we term the motion of matter'.

It is impressive and moving to read this intuitive description of the fundamental ideas of the theory of general relativity written over thirty years before Albert Einstein gave the theory its final form. The subtle relations between