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*Computation with recurrence relations*, by Jet Wimp, Applicable Mathematics Series, Pitman Advanced Publishing Program, Boston, London, Melbourne, 1984, xii + 310 pp., $50.00. ISBN 0-273-08508-5

Recurrence relations occur in a variety of mathematical contexts. They connect a set of elements of a sequence of some type, usually either numbers or functions, such as coefficients in series expansions obtained by undetermined coefficients, moments of weight functions, and members of families of special functions. They can be used either to define the sequence or to produce its elements.

They lead to concise algorithms which are useful for either manual or automatic calculations and can allow great economy in tabulation or approximation. Algorithms based on recurrences are particularly useful for automatic computers because of the compact programs to which they lead, with concomitant economies in memory requirements and in error elimination.

Serious difficulties may be encountered, however, when inexact arithmetic or initial values are used. For example, the modified Bessel functions of the first kind, $I_n(x)$ satisfy the recurrence:

$$y_{n+1}(x) = -(2n/x)y_n(x) + y_{n-1}(x). \tag{1}$$

For $x = 1$, they are positive for all $n$, and decrease monotonously toward 0 as $n$ increases. Using values for $I_0(1) = 1.266065878$ and $I_1(1) = 0.5651591040$, correct to 10 significant digits, and computing $I_2(1), I_3(1), \ldots$ by (1), we find

| $n$ | $I_n(1)$ | $n$ | $I_n(1)$ | $n$ | $I_n(1)$ |
|---|---|---|---|---|---|
| 0 | 0.1266065878 (+1) | 1 | 0.5651591040 (00) | 2 | 0.1357476700 (00) |
| 3 | 0.2216842400 (−1) | 4 | 0.2737126000 (−2) | 5 | 0.2714160000 (−3) |
| 6 | 0.2296600000 (−4) | 7 | −0.4176000000 (−5) | 8 | 0.8143000000 (−4) |
| 9 | −0.1307056000 (−2) | 10 | 0.2360843800 (−1) | 11 | −0.4734758160 (00) |
| 12 | 0.1044007639 (+2) | 13 | −0.2510353092 (+3) | 14 | 0.6537358115 (+4) |

These absurd numerical values are caused by instability in using this recurrence for $I_n(x)$ for increasing $n$. Such difficulties are familiar to numerical mathematicians in many contexts, although they may not be as generally recognized as would be desirable.

Another solution of (1) is $(-1)^n K_n(x)$, where $K_n(x)$ are the modified Bessel functions of the third kind which increase rapidly with increasing $n$. This solution can be evaluated without difficulty starting with values of $K_0(x)$ and $-K_1(x)$ and working forward. Any linear combination of these two solutions is, of course, also a solution of (1). The difficulty in computing the $I_n(x)$ arises because any error in the initial values, or in performing the recurrence, is equivalent to introducing a component of the rapidly growing $(-1)^n K_n(x)$, which soon comes to dominate the results.

This explanation of the instability suggests that one could use (1) to compute sequences of modified Bessel functions of the first kind stably if values of two contiguous large orders, $I_N(x)$ and $I_{N+1}(x)$ say, were known. One rearranges (1) to express $y_{n-1}$ in terms of $y$ and $y_{n+1}$ and computes $I_{N-1}(x), I_{N-2}(x), \ldots$ from it. Any components of the second solution introduced by errors in the initial values will rapidly become insignificant compared to the increasing values of $I_n(x)$. Indeed, as pointed out by J. C. P. Miller [**1952**], one can choose arbitrary values for $y_N(x)$ and $y_{N+1}(x)$, 1 and 0 for example, and find that the lower order elements of the sequence soon become proportional to the corresponding $I_n(x)$. The proportionality constant can be determined by a single value of $I_n(x)$, or even by some normalizing condition such as

$$(2) \qquad I_0(x) - 2I_2(x) + 2I_4(x) - 2I_6(x) + \cdots = 1.$$

For $x = 1.0$ and $N = 14$, this algorithm produces values of $I_n(1)$ with errors no greater than one unit in the tenth significant digit for $n < 11$.

Miller's observation suggested a number of questions:

a. What other sequences of importance in applications can be evaluated to specified precision using backward recurrence?

b. Are there systematic ways of finding recurrences and normalizing conditions for which the Miller algorithm is likely to generate a specified sequence successfully?

c. How can the algorithm be improved to reduce the possibility of overflow and underflow, to improve efficiency, or to provide specified accuracy?

d. What bounds can be set on the errors of the results?

e. What conditions must the recurrence and the normalizing condition satisfy to guarantee convergence?

f. If the algorithm converges for a given recurrence and normalizing condition, what is the sequence to which it converges?

g. Can the algorithm, and the answers to the previous questions, be extended to include inhomogeneous recurrences, recurrences of order greater than two, or nonlinear recurrences?

The three-term recurrence is significantly simpler than higher order ones, and the major results were established by the time of Gautschi's review papers [**1967**, **1972**]. The key to convergence when applying such recurrences in the downward direction is the existence of a minimal solution, $w(n)$, i.e. a solution such that $w(n)/y(n) = o(1)$ for all nontrivial solutions $y(n)$ which are not proportional to $w(n)$. If a minimal solution exists, the sequences $\{y_N(k)\}$ for $0 \leqslant k \leqslant n$ produced by starting with arbitrary $y_N(N)$ and $y_N(N+1)$ (with $N > n$) and computing, successively $y_N(N-1)$, $y_N(N-2), \ldots, y_N(n), \ldots, y_N(0)$, will approach proportionality to the sequence $\{w(k)\}$ as $N$

increases. If the known value of one of the $w(k)$ is used to find the proportionality constant, no further condition is necessary for convergence of the algorithm. If an infinite series must be used, its rate of convergence is also critical.

Demonstrating the existence and identity of a minimal solution for a given recurrence may require a variety of tools drawn from a wide range of classical, if no longer generally familiar, fields of mathematics. For recurrences connecting families of special functions, such as our Bessel function example, one may find two families which form a basis for the general solution, and draw upon the backlog of asymptotic information available. In other cases, the asymptotic theory of difference equations, or the properties of related continued fractions may be helpful.

The three-term recurrence

$$(3) \qquad y(n) + a(n)y(n+1) + b(n)y(n+2) = 0 \qquad n = 0, 1, 2, \ldots$$

is formally equivalent to the continued fraction

$$(4) \qquad \frac{y(n)}{y(n+1)} = -a(n) - \frac{b(n)}{a(n+1) -} \frac{b(n+1)}{a(n+2) -} \cdots$$

for all values of $n$. A theorem of Pincherle shows that (3) has a minimal solution if, and only if, the continued fraction (4) converges. Further, the backward recurrence algorithm may be cast as evaluating finite segments of the continued fraction by starting at the tail and computing successive ratios $y(k)/y(k+1)$. Although formally equivalent to the simple backward recurrence, this algorithm reduces the likelihood of overflow when working with finite numerals. Of course, an additional connection between three-term recurrences and continued fractions comes from the familiar recurrence satisfied by the numerators and the denominators of successive convergents.

Recurrences with more than three terms and inhomogeneous recurrences are harder to treat, because of the more complicated structure of their general solutions—linear combinations of a full basis set of solutions of the homogeneous equation—along with a particular solution, if the recurrence is inhomogeneous. It is thus possible that the desired solution is neither dominant, and thus computable by forward recurrence, nor minimal, so that the Miller algorithm could be applied successfully. Moreover, convergence does not depend on the growth properties of the solutions of the original recurrence, but on those of its adjoint. Algorithms have been devised, however, which, although more complicated, are often successful in generating not only dominant and recessive solutions, but at least some of the intermediate ones. Convergence and error propagation have also been partially analyzed, although the results are difficult to apply.

In this volume Wimp collects (and often extends) algorithms, convergence criteria, and error estimates. He supplements the rigorous analytical treatment of these questions by many examples illustrating both the behavior of the algorithms and the wide variety of problems to which the techniques can be applied. Although many of the examples produce sequences of various hypergeometric functions, others show applications to such problems as determining moments of distributions, finding Gaussian quadrature rules, expressing the solutions of differential equations as power series or as series of orthogonal polynomials, and evaluating and transforming such series.

The last part of the book is concerned with nonlinear multidimensional recurrences and iterations. In contrast with the earlier section, where the emphasis was on the generation of sequences and on the questions of stability and convergence of the backward recurrence as the starting point increased, the principal interest with these recurrences is the behavior of the sequences which are generated as a function of the initial values. Although the strange behavior of sequences which do not converge has attracted considerable recent interest, the field is too new for definitive treatment, and cases in which the sequences do converge to a limit are treated more fully. These include the classical Gauss arithmetic-geometric mean algorithm for the complete elliptic integral, as well as the Borchardt and Bartky algorithms. These are of particular interest both because of the classic nature of the problems which they solve, including the rectification of the lemniscate and ellipse, but also because they provide approaches to evaluating general elliptic functions and integrals which are not hypergeometric and do not satisfy linear differential equations.

In summary, the numerical mathematician concerned with evaluation of special functions will find most of this book of exceptional value, while the mathematician interested in other topics will be introduced to many surprising results, which draw on a wide spectrum of classical mathematical techniques.

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*Modern dimension theory* (second edition, revised and extended), by J. Nagata, Sigma Series in Pure Mathematics, vol. 2, Heldermann-Verlag, Berlin, 1983, 68.00 DM, x + 284 pp. ISBN 3-88538-002-1

Dimension theory is one of the triumphs of point-set topology. When Cantor showed that Euclidean spaces of different dimensions nevertheless admitted one-one correspondences, and Peano showed that this could even happen in a continuous way, the naive ideas about dimension were shattered. Was there even a topological invariant that could be called dimension? Brouwer showed that this was so, at least for Euclidean spaces; but his work did not lead to a satisfactory general theory. The key idea was contained in a remark of