

BOOK REVIEWS

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The H-functions of one and two variables with applications, by H. M. Srivastava, K. C. Gupta, and S. P. Goyal, South Asian Publishers, New Delhi, India, 1982, x + 306 pp., \$25.00.

The generalized hypergeometric function ${}_pF_q$ is defined by the power series

$$(1) \quad {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!}$$

where $(\alpha)_\mu = \Gamma(\alpha + \mu)/\Gamma(\alpha)$ is the Pochhammer symbol. We call x the *variable*, while $a_1, \dots, a_p, b_1, \dots, b_q$ are the *parameters*. Among the particular cases of ${}_pF_q$ are Gauss's hypergeometric function ${}_2F_1$ and the confluent hypergeometric function (Kummer's function) ${}_1F_1$.

The condition $p \leq q + 1$ is necessary to ensure convergence in (1) for $x \neq 0$ (unless the series is terminating). The differential equation for ${}_pF_q$, however, makes sense without this restriction; and in fact one can, by considering suitable integral representations, construct functions that might be termed extensions of generalized hypergeometric functions to the case $p > q + 1$. One such function is MacRobert's *E*-function from 1937, defined by an integral over \mathbf{R}_+^p . More successful, however, are subsequent generalizations in terms of single Mellin-Barnes integrals: Meijer's *G*-function from 1941 and the even more general *H*-function introduced by Fox in 1961. The latter is defined as follows,

$$(2) \quad \begin{cases} H_{p,q}^{m,n} \left[x \left| \begin{matrix} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_L \theta(s) x^s ds, \\ \theta(s) = \frac{\prod_{j=1}^m \Gamma(b_j - \beta_j s) \prod_{j=1}^n \Gamma(1 - a_j + \alpha_j s)}{\prod_{j=-m+1}^q \Gamma(1 - b_j + \beta_j s) \prod_{j=n+1}^p \Gamma(a_j - \alpha_j s)}, \end{cases}$$

where L is a suitable contour in the complex plane. $H_{p,q}^{m,n}$ reduces to a Meijer function when all α 's and β 's are equated to unity. It is noted that Fox's function has *four* types of parameters. Meijer's and Fox's functions are treated in detail in two recent monographs by Mathai and Saxena [1, 2].

In principle, there are no difficulties in constructing hypergeometric functions of N variables by means of multiple sums or multiple Mellin-Barnes integrals. However, the number of possible types of parameters increases considerably with N in accordance with the possible ways of associating parameters with variables. Thus, any 'general' hypergeometric function of N variables would be a notational monster; and all investigations of multiple hypergeometric functions are restricted to suitable subclasses.

In particular, the two-variable generalization of Fox's function is defined by a double Mellin-Barnes integral of the form

$$\int_{L_1} \int_{L_2} \theta(s) \psi(t) \phi(s, t) x^s y^t ds dt,$$

where $\psi(t)$ has the same structure as $\theta(s)$, cf. (2), while $\phi(s, t)$ contains $\Gamma(1 - a_j + \alpha_j s + A_j t)$, etc.

This function has been extensively studied in recent years, and these developments are presented in the monograph under review, which "aims at familiarizing workers in the fields of special functions and allied topics of mathematical analysis and applicable mathematics with the theoretic and applicative aspects of the H -functions of one and two variables." The contents of the book may be briefly outlined as follows.

An account of the historical background in Chapter 1 is followed by four chapters on the H -function of one variable. Chapter 2 deals with the definition, special cases and various elementary results (asymptotic expansions, contiguous relations, inequalities, differentiation and summation formulas, etc.). In Chapter 3, integral equations, Mellin convolutions and fractional integral operators are discussed. Chapter 4 presents a general integral transform with an H -function kernel; an inversion formula and other useful theorems are given. Also, Fourier kernels involving the H -functions are considered. Chapter 5 contains a number of single and multiple integrals involving products of one or several H -functions and simpler functions; moreover, some expansions of H -functions in series of orthogonal functions are considered.

The four following chapters are devoted to the H -function of two variables. Chapter 6 gives the definition, special cases and a number of elementary properties (e.g., transformations, derivations, asymptotic behavior). Chapter 7 contains contiguous recurrence relations, some finite series, and some unified summation formulas of a general nature. In Chapter 8, some interesting single and multiple integrals whose integrands involve H -functions of two variables are studied. Chapter 9 contains expansions in series of orthogonal functions (e.g., Bessel functions) and in series of hypergeometric polynomials. Moreover, some generating functions are obtained.

Chapter 10 deals with the applications of the H -functions in statistics and some branches of mathematical physics. Also, the H -function with a matrix argument is considered. In Chapter 11, the authors study a general two-dimensional integral transform whose kernel involves the H -function of two variables. An inversion theorem and other interesting results are given; also, the

connections with classical transforms are discussed. The final Chapter 12 is a study of convolution integral equations involving the H -function of two variables.

The Appendix lists useful formulas for special functions of one and several variables, along with a brief account of an H -function of several variables. The bibliography contains more than one thousand references.

The wealth of material is, in the reviewer's opinion, sensibly selected and arranged. Proofs are not necessarily given; and in some cases the authors deal summarily with large families of formulas by merely deriving the *numbers* of such formulas. Moreover, it should be noted that H -symbols are satisfactorily printed, in spite of their typographical complexity.

The book, providing a comprehensive account of a subject widely scattered in the literature, will be of value to all researchers and students in the field of special functions.

REFERENCES

1. A. M. Mathai and R. K. Saxena, *Generalized hypergeometric functions with applications in statistics and physical sciences*, Lecture Notes in Math., vol. 348, Springer-Verlag, Berlin, Heidelberg and New York, 1973.

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The theory of topological semigroups, by J. H. Carruth, J. A. Hildebrant, and R. J. Koch, Pure and Applied Mathematics, Marcel Dekker, Inc., 1983, vi + 244 pp., \$34.75. ISBN 0-8-8247-1795-3

BACKGROUND. Very few subject areas in mathematics can assign their creation to a single individual. Topological semigroups is one of the exceptions. Alexander Doniphan Wallace is universally acknowledged to have fostered the idea of studying continuous, associative multiplications on Hausdorff spaces, and no practitioner could wish for a more colorful father of the subject.

An excellent teacher of graduate students and an able wordsmith, Wallace was the natural choice of his descendants to chronicle the growth of the subject. Regrettably, the continuing press of administrative duties prevented this project from every being seriously begun. An early hint of what might have been is contained in the Bulletin article [4] that summarized his 1955 address to the Society—one of the most cited sources ever in American mathematics publishing.

By the early 1960s, the only book in print on the subject (besides Wallace's jealously guarded course notes) was a Centrum tract by Paalman-De Miranda [3], an effort clearly not intended to serve as a text. The appearance of the first volume of Clifford and Preston's work in algebraic semigroups [1] made the