RESEARCH ANNOUNCEMENTS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 14, Number 1, January 1986

DEFORMATION SPACES ASSOCIATED TO COMPACT HYPERBOLIC MANIFOLDS

BY DENNIS JOHNSON AND JOHN J. MILLSON

Recently, there has been considerable interest in spaces of locally homogeneous (or geometric) structures on smooth manifolds, motivated by Thurston [**6**, **7**]. If M is a smooth manifold, we will let $\mathcal{C}(M)$ denote the space of conformal structures (with marking) on M and $\mathcal{P}(M)$ the space of projective structures (with marking) on M. Since these spaces are a measure of the complexity of the fundamental group, it makes sense to consider the case in which M admits a hyperbolic structure. We note that in case n, the dimension of M, is strictly greater than 2, this hyperbolic structure is unique by the Mostow Rigidity Theorem. Hence, $\mathcal{C}(M)$ and $\mathcal{P}(M)$ each have a finite number of distinguished points, the conformal and projective structures associated to the hyperbolic structure with the various possible markings of $\pi_1(M)$.

In order to study $\mathcal{C}(M)$ and $\mathcal{P}(M)$, it is convenient to replace $\mathcal{C}(M)$ and $\mathcal{P}(M)$ with the space of conjugacy classes of representations of Γ, the fundamental group of M, into the automorphism groups $SO(n+1,1)$ and $PGL_{n+1}(\mathbf{R})$ of the model spaces S^n and $\mathbf{RP}^n$. This is possible because of a general result of Lok [**2**].

Let $\mathcal{S}(M)$ be a space of (marked) locally homogeneous structures modelled on a homogeneous space $X = G/H$ with G a semisimple linear algebraic group. Given a structure $s \in \mathcal{S}(M)$, by continuing coordinate charts around elements of Γ, we obtain the holonomy representation ρ of Γ into G and a map

$$\text{hol}\colon \mathcal{S}(M) \to \text{Hom}(\Gamma, G)/G$$

defined so that hol(s) is the orbit of ρ under conjugation by G. Then Theorem 1.11 of Lok [**2**] states that hol is an open map which lifts to a local homeomorphism from the space of (G,X)-developments to $\text{Hom}(\Gamma, G)$. We will refer to this result as the "Holonomy Theorem". Unfortunately hol is not necessarily a local homeomorphism. To deal with this point we say that a representation ρ of Γ is *stable* if the image of ρ is not contained in a parabolic subgroup

Received by the editors September 20, 1984 and, in revised form, July 29, 1985.
1980 *Mathematics Subject Classification*. Primary 22E40.

of the complexification of G. The set S of stable representations is open in $\mathrm{Hom}(\Gamma, G)$ and the action of G on S admits slices through every point of S. It follows from the Holonomy Theorem that if $\rho = \mathrm{hol}(s)$ is stable there exist neighbourhoods U of s in $\mathcal{S}(M)$ and V of ρ in $\mathrm{Hom}(\Gamma, G)/G$, finite groups H_1 and H_2 with $H_1 \subset H_2$ (the isotropy subgroups of s and ρ) and finite quotient mappings $U = \tilde{U}/H_1$, $V = \tilde{V}/H_2$ such that hol lifts to a homeomorphism from $\tilde{U}$ to $\tilde{V}$. Now, if M is a marked hyperbolic n-manifold, we have the uniformization representation $\rho_0\colon \Gamma \to SO(n,1)$. The distinguished conformal and projective structures associated to the hyperbolic structure correspond under hol to the representations obtained by composing ρ_0 with natural representations $SO(n,1) \to SO(n+1,1)$ and $SO(n,1) \to PGL_{n+1}(\mathbf{R})$. We will denote the resulting compositions again by ρ_0. Since the image groups $\rho_0(\Gamma)$ have infinite covolume, it is possible that they will have nontrivial deformations.

In order to state our theorem to this effect, we define r to be the number of connected hypersurfaces in a maximal collection of disjoint, nonsingular, two-sided (i.e., having trivial normal bundle), totally geodesic hypersurfaces in M. For example, in a hyperbolic surface of genus g it is well known that $r = 3g - 3$. We remark that for each n, r may be made arbitrarily large by choosing suitable congruence covers of the hyperbolic n-manifolds constructed from unit groups of quadratic forms over totally real number fields. We have the following theorem.

THEOREM 1. *The spaces*

$$\mathrm{Hom}(\Gamma, SO(n+1,1))/SO(n+1,1)$$

and

$$\mathrm{Hom}(\Gamma, PGL_{n+1}(\mathbf{R}))/PGL_{n+1}(\mathbf{R})$$

each have dimension greater that r.

COROLLARY. *The spaces $\mathcal{C}(M)$ and $\mathcal{P}(M)$ have dimension greater than or equal to r.*

To state our second theorem, we let G denote either $SO(n+1,1)$ or $PGL_{n+1}(\mathbf{R})$ and $\mathbf{G}$ denote the complexification of G. Then $\mathrm{Hom}(\Gamma, G)$ is a real algebraic set. Also since $\mathbf{G}$ is reductive, by Newstead [**3**] there is a quotient variety of $\mathrm{Hom}(\Gamma, \mathbf{G})$ by $\mathbf{G}$ which we denote $X(\Gamma, \mathbf{G})$. This variety is defined over $\mathbf{R}$, we denote its real points by $X(\Gamma, G)$. We have the following theorem for suitable congruence subgroups of the arithmetic groups mentioned above. We assume $n \geq 4$.

THEOREM 2. (i) *$\mathrm{Hom}(\Gamma, G)$ has a nonisolated singularity at ρ_0. In particular, there exist **irreducible** representations of Γ in G which are singular points.*

(ii) *$X(\Gamma, G)$ has a nonisolated singularity at the class of ρ_0. In particular, there exist classes of **irreducible** representations of Γ in G which are singular points.*

REMARK. One should be able to show, by using the Kuranishi theory (see Kodaira-Morrow [**1**, Chapter IV]), that $\mathcal{C}(M)$ and $\mathcal{P}(M)$ are real analytic spaces and that hol is a local equivalence. Consequently, $\mathcal{C}(M)$ and $\mathcal{P}(M)$ would have singularities by Theorem 2.

We now discuss the proofs of Theorems 1 and 2. Theorem 1 is proved by using an algebraic version of Thurston's bending deformation–for the geometric version see Sullivan [**5**]. Let $\mathcal{H}$ be a collection of r disjoint, two-sided, totally geodesic hypersurfaces in M. Let $\tilde{\mathcal{H}}$ be the complete inverse image of the collection $\mathcal{H}$ in $\tilde{M}$, the universal cover of M. We may associate a graph X to $\tilde{\mathcal{H}}$ and a graph Y to $\mathcal{H}$ such that X is a tree, Γ acts on X and the quotient graph of X is Y. These graphs are obtained by taking a vertex for each connected component of the complement of the elements of $\mathcal{H}$ in M (or $\tilde{\mathcal{H}}$ in $\tilde{M}$), an edge e for each hypersurface, taking for the terminus of e the vertex corresponding to the component containing the positive side of the hypersurface N_e corresponding to e and for the origin of e the vertex corresponding to the component containing the negative side of N_e. Then, by Serre [**4**], we find that Γ is the fundamental group of a graph of groups and that the edge group Γ_e is the fundamental group of the hypersurface N_e. The main point is that because N_e is totally geodesic the centralizer of $\rho_0(\Gamma_e)$ contains a one-parameter group $a_t(e)$. We use the r one-parameter groups $\{a_t(e)\colon e \text{ an edge of } Y\}$ to construct an r-parameter deformation of ρ_0. This is possible because the relations defining the fundamental group of a graph of groups are very simple.

We now discuss the proof of Theorem 2. Suppose that M contains two totally geodesic hypersurfaces which intersect in a codimension 2, totally geodesic submanifold. From the previous discussion, N_1 and N_2 give rise to curves ρ_{t_1} and ρ_{t_2} of representations. As is well known, we may identify the tangent vectors $\dot\rho_1$ and $\dot\rho_2$ to these curves with crossed homomorphisms from Γ to $\mathfrak{g}$, the Lie algebra of G. We may ask if the crossed homomorphism $\dot\rho_1 + \dot\rho_2$ is also tangent to a curve in $\mathrm{Hom}(\Gamma, G)$. To answer this, we associate to $\dot\rho_1 + \dot\rho_2$ an Eilenberg-Mac Lane 2-cocycle on Γ with values in $\mathfrak{g}$, denoted $[\dot\rho_1 + \dot\rho_2, \dot\rho_1 + \dot\rho_2]$, by the formula

$$[\dot\rho_1 + \dot\rho_2, \dot\rho_1 + \dot\rho_2](\gamma, \delta) = [(\dot\rho_1 + \dot\rho_2)(\gamma), \mathrm{Ad}\ \rho_0(\gamma)((\dot\rho_1 + \dot\rho_2)(\delta))].$$

Here the bracket on the right denotes the Lie bracket in $\mathfrak{g}$, and γ and δ are in Γ. It is easy to see that a necessary condition for $\dot\rho_1 + \dot\rho_2$ to be tangent to a curve in $\mathrm{Hom}(\Gamma, G)$ is that $[\dot\rho_1 + \dot\rho_2, \dot\rho_1 + \dot\rho_2]$ be an Eilenberg-Mac Lane coboundary. We can construct examples where this is not the case. This proves that $\mathrm{Hom}(\Gamma, G)$ is singular, since its tangent cone is not a linear space. To obtain nonisolated singularities, we find a totally geodesic hypersurface N_3 disjoint from N_1 and N_2. Then the curve in $\mathrm{Hom}(\Gamma, G)$ obtained by bending along N_3 consists entirely of singular points (bending along N_3 does not change the restriction of ρ_0 to the fundamental groups of N_1 and N_2).

We should give some indications of how to compute the cup-product $[\dot\rho_1 + \dot\rho_2, \dot\rho_1 + \dot\rho_2]$. It is immediate that $[\dot\rho_1 + \dot\rho_2, \dot\rho_1 + \dot\rho_2]$ is cohomologous to $2[\dot\rho_1, \dot\rho_2]$. We then calculate the dual homology classes (with coefficients in $\mathfrak{g}$) to $\dot\rho_1$ and $\dot\rho_2$ and compute the intersection product of representing dual cycles by geometry. This is possible because these dual cycles have a very simple description as follows. The fundamental group of a totally geodesic hypersurface N in M has a unique invariant line L in $\mathfrak{g}$. Choose any ω in L. Then the fundamental class of N may be combined with ω to obtain a cycle with coefficients denoted $N \otimes \omega$ (in fact N does not have to be orientable, only two-sided). We can find invariants ω_1 and ω_2 so that $\dot{\rho}_1$ is dual to $N_1 \otimes \omega_1$ and $\dot{\rho}_2$ is dual to $N_2 \otimes \omega_2$. The calculation then reduces to computing $N_1 \cap N_2$, a tractable calculation since N_1 and N_2 are totally geodesic.

We conclude this announcement by thanking S. Y. Cheng, Larry Lok, John Morgan and especially Bill Goldman for helpful discussions. Our paper would never have been written were it not for Bill Thurston's idea of bending a Fuchsian group. Our paper will appear in *Discrete groups in geometry and analysis*, the proceedings of a conference held at Yale University in honor of G. D. Mostow on his sixtieth birthday.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT LOS ANGELES, LOS ANGELES, CALIFORNIA