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HOMOTOPY CLASSES IN SOBOLEV SPACES AND ENERGY MINIMIZING MAPS

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Let M and N be compact Riemannian manifolds. The *energy* of a lipschitz map $f\colon M \to N$ is $\int_M |Df|^2$ (where $|Df(x)|^2 = \sum |\partial f/\partial x_i|^2$ if $x_1, \ldots, x_m$ are normal coordinates for M at x). Mappings for which the first variation of energy vanishes are called *harmonic*. The identity map from M to M is always harmonic, but it may be homotopic to mappings of less energy. For instance, the identity map on S^3 is homotopic to mappings of arbitrarily small energy (namely, conformal maps that pull points from the North Pole toward the South Pole). That suggests the question: For which manifolds M is the identity map homotopic to maps of arbitrarily small energy? In this paper we give the simple answer: Those M such that $\pi_1(M)$ and $\pi_2(M)$ are both trivial. More generally, we consider energy functionals like $\Phi(f) = \int_M |Df|^p$ and ask:

(1) When is the infimum of $\Phi(f)$ in some homotopy class of mappings $f\colon M \to N$ nonzero?

(2) When is the infimum of $\Phi(g)$ (among maps satisfying some homotopy condition) actually attained?

To answer such questions, it is convenient to regard N as isometrically embedded in a euclidean space $\mathbf{R}^\nu$ and to work with the Sobolev norm

$$\|f\|_{1,p} = \left(\int_M |f|^p\right)^{1/p} + \left(\int_M |Df|^p\right)^{1/p}$$

(where $f\colon M \to \mathbf{R}^\nu$ has distribution derivative Df) and with the associated Sobolev spaces,

$$L^{1,p}(M,N) = \{f\colon M \to \mathbf{R}^\nu \mid f(x) \in N \text{ for a.e. } x, \text{ and } \|f\|_{1,p} < \infty\}$$

and

$$W^{1,p}(M,N) = \text{the closure of } \{\text{lipschitz maps } f\colon M \to N\} \text{ in } L^{1,p}(M,N).$$

Say that two continuous maps $f, g\colon M \to N$ are *k-homotopic* (or have the same *k-homotopy type*) if their restrictions to the k-dimensional skeleton of some triangulation of M are homotopic. We have the following theorem about $W^{1,p}(M,N)$ (where $[p]$ is the integer part of p).

THEOREM 1. *Two lipschitz maps are in the same connected component of $W^{1,p}(M,N)$ if and only if they are $[p]$-homotopic. Consequently every map in $W^{1,p}(M,N)$ has a well-defined $[p]$-homotopy type. Furthermore, the set of*

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lipschitz maps homotopic to a given map f is dense (with respect to $\|\cdot\|_{1,p}$) in the connected component containing f.

As a corollary we have the answer to (1).

COROLLARY. *The infimum of $\Phi(g)$ among lipschitz maps $g: M \to N$ homotopic to a given lipschitz map $f: M \to N$ is equal to the infimum of $\Phi(g)$ among all lipschitz maps that are merely $[p]$-homotopic to f. In particular, the infimum is 0 if and only if the restriction of f to the $[p]$-skeleton of M is homotopically trivial.*

The space $W^{1,p}(M, N)$ is not, however, suitable for studying existence questions such as (2) because it lacks nice compactness properties. In $L^{1,p}(M, N)$, on the other hand, closed bounded sets are compact in the weak topology. We have

THEOREM 2. *Every $f \in L^{1,p}(M, N)$ has a well-defined $[p-1]$-homotopy type. If $f_i \in L^{1,p}(M, N)$ is a $\|\cdot\|_{1,p}$-bounded sequence of functions with a given $[p-1]$-homotopy type, and if f_i converges weakly to f, then f has the same $[p-1]$-homotopy type.*

This gives the answer to (2).

COROLLARY. *The infimum of $\Phi(g)$ among all maps $g \in L^{1,p}(M, N)$ with a given $[p-1]$-homotopy type is attained.*

In case $p = 2$, then $\Phi(g)$ is the ordinary energy of g, and the minimizing map g is locally energy minimizing in the sense studied by Schoen and Uhlenbeck [**SU1,2**]. By combining the above existence result with their regularity theorems, we obtain

THEOREM 3. *In every 1-homotopy class of mappings in $L^{1,2}(M, N)$, there is a map g of least energy. Such a map is a smooth harmonic map except on a closed set $K \subset M$ of Hausdorff dimension $\leq \dim(M) - 3$.*

Furthermore, if N has negative sectional curvatures or if $\dim M = 3$ and N is any surface other than S^2 or $\mathbf{RP}^2$, then the map is completely regular. Since in these cases N has a contractible covering space, the homotopy type of g is determined by its 1-homotopy type. Consequently

THEOREM 4. *If (1) N has negative sectional curvatures, or*
(2) $\dim M = 3$ and N is a surface other than S^2 or $\mathbf{RP}^2$, then every homotopy class of mappings from M to N contains a smooth map g of least energy.

The main tools in the proofs are: a deformation procedure analogous to the Federer-Fleming one [**F**, 4.2.9], versions of the Poincaré and Sobolev inequalities that hold for polyhedral complexes (such as the k-skeleton of M), and the homotopy extension theorem. All of the results generalize in the expected way to manifolds M with boundary.

Some special cases of these results were known previously: see [**W2**] and [**W3**] for details and references. Also, Theorem 4(1) was originally proved in a different way by Eells and Sampson [**ES**]. The analogous questions for area instead of energy are studied in [**SU**, **SY**] (when $\dim M = 2$) and [**W1**] (when $\dim M > 2$).

In [**EL**, II.2.4-5] it is pointed out that Theorem 4(2) follows from the case $p = 2$ of Theorem 2. However, it seems that no proof (even in that case) has been published (though Schoen and Yau [**SY**] gave a proof when $\dim M = p$).

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