BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 13, Number 2, October 1985

L^2 HARMONIC FORMS AND A CONJECTURE OF DODZIUK-SINGER

BY MICHAEL T. ANDERSON[1]

Let M^n be a complete simply connected Riemannian manifold of sectional curvature K_M satisfying $-a^2 \leq K_M \leq -1$, $a \geq 1$. Let $\mathcal{H}_2^p(M^n)$ denote the space of L^2 harmonic p-forms on M, i.e. p-forms $\omega \in \Lambda^p(M^n)$ such that

$$\Delta\omega = 0, \qquad \int_{M^n} \omega \wedge *\omega = \int_{M^n} |\omega|^2 \, dV < \infty.$$

It is clear that $\mathcal{H}_2^p M^n$ is naturally isomorphic to $\mathcal{H}_2^{n-p}$ under the Hodge $*$ operator, and $\mathcal{H}_2^0(M^n) = 0$. Further, it is known [**2**] that $\mathcal{H}_2^*(M^n)$ naturally injects into the L^2-cohomology of M^n. Dodziuk and Singer (see [**3**, **4** and **6**]) have conjectured that $\mathcal{H}_2^p(M^n) = 0$ if $p \neq n/2$ and $\dim \mathcal{H}_2^{n/2} = \infty$ if n is even. An affirmative solution of this conjecture implies, by means of the L^2 index theorem for regular covers of Atiyah [**1**], a positive solution of the well-known Hopf Conjecture: If M^{2m} is a compact manifold of negative sectional curvature, then $(-1)^m \chi(M^n) > 0$.

Dodziuk [**3**] has proved the L^2 form conjecture for rotationally symmetric metrics—in particular for the space forms $H^n(-a^2)$ of curvature $-a^2$. Donnelly and Xavier [**5**] have recently obtained results in case the curvature of M^n is sufficiently pinched: They show $\mathcal{H}_2^p(M^n) = 0$ if $0 < p < (n-1)/2$ and $a < (n-1)/2p$.

In this note, we outline the construction of counterexamples to the L^2 form conjecture, in every dimension and degree except the middle. Our main result is

THEOREM. *For any $n \geq 2$, $0 < p < n$ and $a > |n - 2p|$, with $a \geq 1$, there exist complete simply connected Riemannian manifolds M^n with*

$$-a^2 \leq K_M \leq -1$$

such that $\dim \mathcal{H}_2^p(M^n) = \infty$.

These manifolds have large isometry groups, $I(M) = O(2p-1,1) \times O(n-2p+1)$: the principal orbits have codimension $n - 2p$. However, $I(M)$ does not have discrete cocompact subgroups and thus M^n cannot be used to construct counterexamples to the Hopf conjecture. There are quotients of the topological form $\overline{M}^{2p-1} \times R^{n-2p+1}$, where $\overline{M}^{2p-1}$ is a compact manifold of curvature -1.

Received by the editors January 29, 1985.
1980 *Mathematics Subject Classification*. Primary 58C35, 58G05, 53C20.

[1]NSF Mathematical Sciences Postdoctoral Fellow.

OUTLINE OF CONSTRUCTION. We define the manifolds M^n to be warped products

$$M^n = H^{2p}(-a^2) \times_f S^{n-2p}(1),$$

where $S^{n-2p}(1)$ is the space form of curvature $+1$ and $f\colon H^{2p}(-a^2) \to R$, $f(x) = \sinh s(x)$, where s is the distance to a fixed totally geodesic hyperplane $H^{2p-1} \subset H^{2p}(-a^2)$. The metric on M^n is given by

$$ds^2 = ds^2_{H^{2p}(-a^2)} + f^2\, ds^2_{S^{n-2p}(1)}.$$

One easily verifies that (M^n, ds^2) is a complete Riemannian manifold, diffeomorphic to R^n.

(i) CURVATURE OF M: Let $\{X_i\}$ be a local orthonormal framing of $H^{2p}(-a^2)$ by eigenvectors of $D^2 f$ and $\{V_j\}$ a local orthonormal framing of $S^{n-2p}(1)$. One may show that the family of 2-forms $\{X_i \wedge X_j\}$, $\{X_i \wedge V_j\}$, $\{V_i \wedge V_j\}$ diagonalizes the curvature operator $\mathcal{R}\colon \Lambda^2(TM) \hookrightarrow$ with corresponding sectional curvatures $-a^2$, $-a \coth s \cdot \tanh as$, -1. In particular, the sectional curvatures of M lie in the range $[-a^2, -1]$.

(ii) HARMONIC FORMS ON M: Let $\omega \in \Lambda^p(H^{2p}(-a^2))$ be invariant under reflection through H^{2p-1} and extend ω to M by defining it to be invariant under the isometric $SO(n-2p+1)$ action on M. One computes that

$$\Delta_M \omega = \Delta_{H^{2p}} \omega + (-1)^p [d \circ \iota_F - \iota_F \circ d]\omega \tag{1}$$

where $F = (n-2p)\, df/f$ is the negative of the mean curvature of $S^{n-2p} \subset M^n$ and ι denotes interior multiplication. We outline a procedure reducing the case of general p to $p = 1$. First, note the identity

$$H^{2p}(-a^2) = H^2(-a^2) \times_g H^{2p-2}(-a^2),$$

where $g\colon H^2(-a^2) \to R$, $g(x) = \cosh ar(x)$, r is the distance function to a fixed point $0 \in H^2(-a^2)$. Further, under this decomposition, F is tangent to the $H^2(-a^2)$ factors. Set

$$\omega = \phi \wedge \eta, \qquad \phi \in \Lambda^1(H^2(-a^2)), \qquad \eta \in \Lambda^{p-1}(H^{2p-2}(-a^2)).$$

If η is any harmonic $(p-1)$-form on $H^{2p-2}(-a^2)$, then ω satisfies (1) if and only if

$$\Delta\phi - [d \circ \iota_F - \iota_F \circ d]\phi = 0 \quad \text{on } \Lambda^1(H^2(-a^2)). \tag{2}$$

To study the solutions of (2), set $\phi = du$ and use the conformal equivalence of $H^2(-a^2)$ with $\Omega = \{(x,\theta)\colon x \in R,\ \theta \in (-\pi/2, \pi/2)\}$ to obtain the equivalent equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial \theta^2} + \phi(\theta)\frac{\partial u}{\partial \theta} = 0, \tag{3}$$

where $\phi(\theta) = (1/f_1)(\partial f_1/\partial\theta)$ and $f_1 = f|_{H^2(-a^2)}$: explicitly,

$$f_1 = f_1(\theta) = \frac{1}{2}\left[\frac{\alpha^{1/a} - \beta^{1/a}}{\cos^{1/a}\theta}\right]$$

where $\alpha = 1 + \sin\theta$, $\beta = 1 - \sin\theta$. Note that ϕ degenerates on $\partial\Omega$. We may assume, without loss of generality, that $(n-2p) > 0$, so $\phi > 0$.

It is now quite straightforward to verify that (3) has solutions, smooth up to $\partial\Omega$. If we conformally identify $H^2(-a^2)$ with $B^2(1)$ with the flat metric, one may produce an infinite-dimensional space of solutions of (3) with $|du|_\infty < 1$.

(iii) L^2 ESTIMATE: First, we recall that $|\omega|^2 = \int \omega \wedge *\omega$ is a conformal invariant for forms in the middle dimension. For ω as above, we have

$$\begin{aligned} \int_{M^n} |\omega|^2 &= \int_{H^{2p}\times S^{n-2p}} |\omega|^2 f^{n-2p}\, dV_H\, dV_S \\ &= \operatorname{vol}(S) \int_{H^2\times H^{2p-2}} |\phi|^2\, |\eta|^2 f^{n-2p}\, dV_{H^2}\, dV_{H^{2p-2}} \\ &\le \operatorname{vol} S^{n-2p} \cdot \operatorname{vol} B^{2p-2}(1) \cdot \int_{B^2} f^{n-2p}\, dV_B, \end{aligned}$$

where we have used the conformal equivalence of $H^k(-a^2)$ with $B^k(1)$, $k = 2, 2p-2$ and assumed that η is a harmonic $(p-1)$-form with $|\eta|_\infty \le 1$ with respect to the flat metric on $B^{2p-2}(1)$, e.g. $\eta = (1/(p-1)!)\, dx_1 \wedge \cdots \wedge dx_{p-1}$. One checks that

$$\int_{B^2(1)} f^{n-2p}\, dV < c \cdot \int_0^{\pi/2} \cos^{-(n-2p)/a}\theta\, d\theta,$$

so that if $(n-2p)/a < 1$, one has $\int_{M^n} |\omega|^2 < \infty$.

Further discussion and examples will appear elsewhere.

REFERENCES

1. M. F. Atiyah, *Elliptic operators, discrete groups and Von Neumann algebras*, Asterisque **32–33** (1976), 43–72.
2. J. Cheeger, *On the Hodge theory of Riemannian pseudomanifolds*, Proc. Sympos. Pure Math. vol. 36, Amer. Math. Soc., Providence, R.I., 1980, pp. 91–146.
3. J. Dodziuk, *L^2 harmonic forms on rotationally symmetric Riemannian manifolds*, Proc. Amer. Math. Soc. **77** (1979), 395–400.
4. ——, *L^2 harmonic forms on complete manifolds*, Seminar on Differential Geometry, ed. S. T. Yau, Ann. of Math. Studies, no. 102, Princeton Univ. Press, Princeton, N.J., 1982.
5. H. Donnelly and F. Xavier, *On the differential form spectrum of negatively curved Riemannian manifolds*, Amer. J. Math. **106** (1984), 169–185.
6. S. T. Yau, *Problem section*, Seminar on Differential geometry, Ann. of Math. Studies, no. 102, Princeton Univ. Press, Princeton, N.J., 1982.

INSTITUT DES HAUTES ETUDES SCIENTIFIQUES, 35, ROUTE DE CHARTRES, 91440 BURES-SUR-YVETTE, FRANCE

Current address: Department of Mathematics, California Institute of Technology, Pasadena, California 91125